Extended Tripod Notations

Graham Breed October MMXI

I introduced tripod notation in `http://x31eq.com/magic/tripod.pdf`. There, I described the system in terms of 9-limit intervals (a prime-limit of 7) using Marvel temperament. Here, I'm going to extend it to cover the prime-limit of 13. Because there are different mappings for Marvel temperaments that might apply at these dizzying heights, I'll outline all the ones that are likely to be useful.

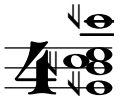
Although the focus here will be on tripod notation, and so not relevant for the overwhelming majority of mankind who won't be using it, I will consider details of Marvel extensions that may be of slightly wider interest. I'll show diagrams mapping ratios to 5-limit lattices and, later on for those brave souls who remain, I'll also describe the mappings in terms of matrix algebra.

The primes 11 and 13 are interesting because they bring more exotic intervals into the harmonic palette. Because of the temperament, these intervals are already available as combinations of the core 9-limit consonances. Talking about higher limits means they can be thought of as coherent harmonic units instead of arbitrary collections of intervals. The tuning can also be optimized to bring out the new intervals but I won't be explaining that here.

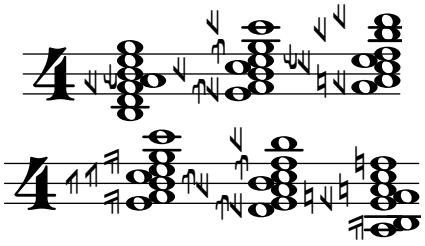
Eleven-Limit Extensions

Our first step up from 9 is to 11. The new 11-limit harmonies are less obviously consonant than the 9-limit, but they're still good enough that they can be improved by paying attention to the tuning. Whereas 9-limit chords will tend to sound like differently-tuned sonorities from chromatic music, intervals of 11 (11-limit intervals that include a factor of 11) tend to fall in the gaps between the steps of the chromatic scale — giving neutral intervals or quartertones — and so emancipate the enharmonic genus.

There's a step in the tripod scale between the feet that we could call a neutral second. Let's call it a big toe. How about approximating it as 12:11 or 11:10? Call it 12:11 and the result is 11-limit Marvel: the mainstream extension of the 7-limit geometry that tripod notation is based on. An 8:10:11:12:16 chord will look like this:



The 9 and 14 of that chord both need inch or comma shifts to write, because they belong on the wrong feet. The chord can't be transposed without these shifts because it depends on 11 and 12 being on different feet. Here are some transpositions of 8:9:10:11:12:14:16 using inch shifts



The inch symbol is new. It's more sagitally correct than the one I used before, which was biased towards Magic temperament.

Figure 1 shows 11-limit ratios[1] on a 5-limit (Didymic) lattice with approximations

[1] I added a few ratios beyond the odd-limit of 11. You don't have to pay extra for these pitches. So don't complain that these lattices don't only show the odd limits they're labeled for.

Figure 1: 11-limit ratios on the Marvel lattice

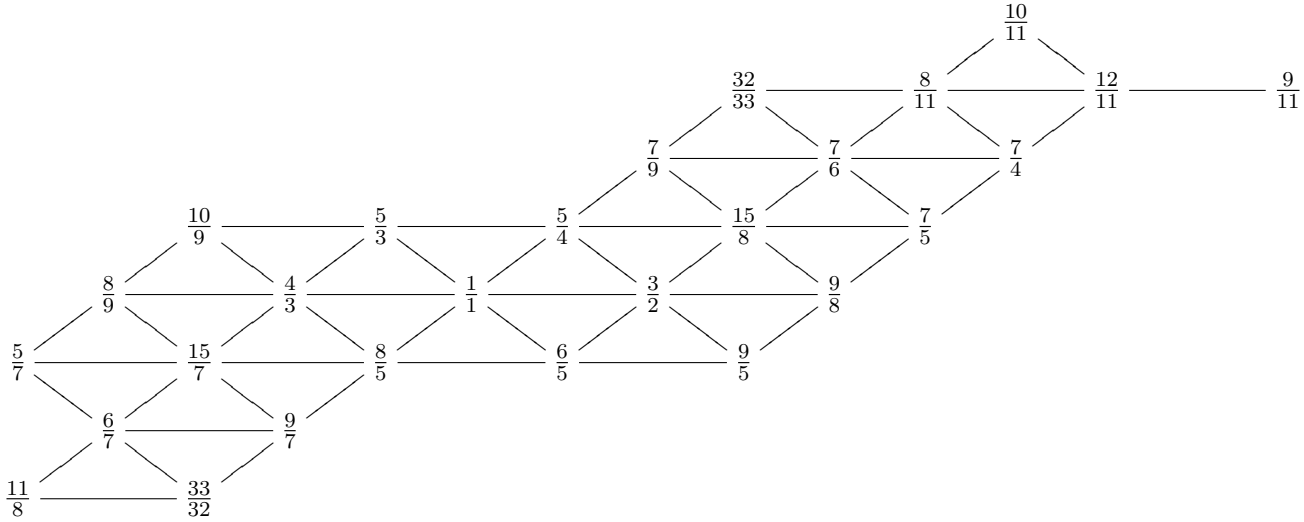
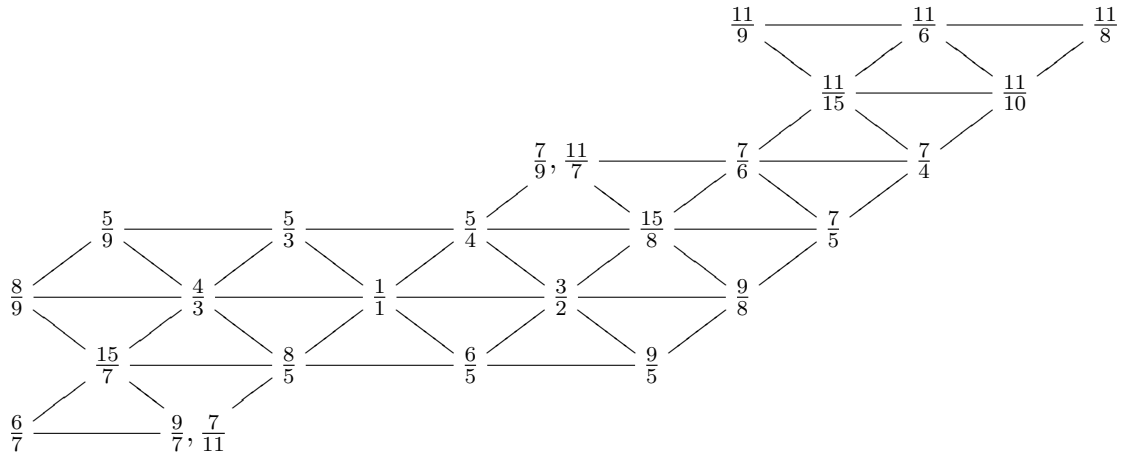
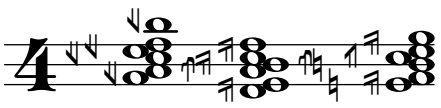


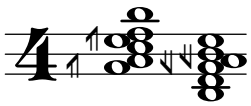
Figure 2: 11-limit ratios on the Minerva lattice



according to Marvel temperament. You can see that the intervals of 11 tend to be neighbors of 9-limit intervals with every 11-limit interval having a unique position on the lattice, and so a unique spelling in any Marvel notation. The 11-limit portion of the lattice tends to slant to the right, in a way consistent with the major-third basis of tripod notation.

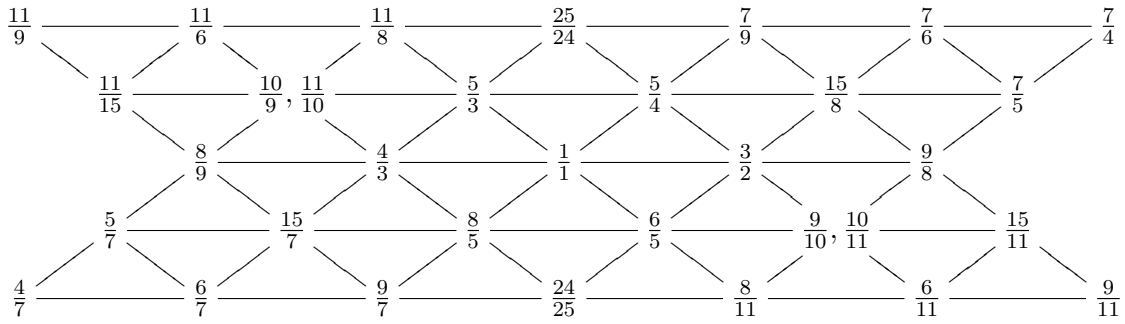
The other tripod-friendly approximation of the big toe is 11:10. That leads to something called Minerva. It makes 8:9:10:11:12:14 chords look like this:





Some of these chords work without inch shifts because Minerva is a slightly simpler mapping than Marvel. You can see how Minerva approximations map to the 5-limit lattice in Figure 2. Most ratios are distinct, but 9:7 and 14:11 are merged. The slope is about the same as for Marvel. The most important difference is that the numerators with 11 are in the top right for Minerva, but the bottom left for Marvel. This is why Minerva counts as simpler. A 4:5:6:7:9:11 chord only needs half of the Minerva lattice but spreads across

Figure 3: 11-limit ratios on the Apollo lattice

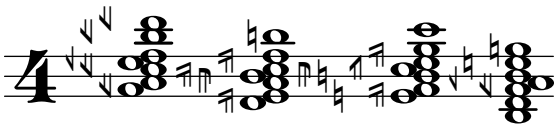


the Marvel lattice.

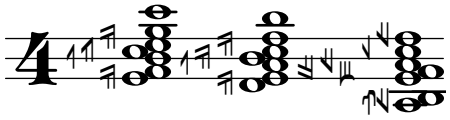
Putting the numerators of 11 on different sides of the lattice is also important when you have a fixed set of pitches to work with. That may be on a keyboard, or where you want to economize on alteration symbols in notation. When you add the 11 to a 9-limit chord, you might not get a choice of Marvel or Minerva, but one of them should work. You may decide to mix the spellings in the same piece. As regular temperaments, they optimize differently, and Marvel is more accurate overall (merging 9:7 and 14:11 has a price). If the tuning has to be fixed, you can optimize it for Marvel, and if any Minerva chords aren't too prominent the relative mistuning probably won't matter.

Tripod notation was designed for Magic temperaments. Note that the 11-limit Marvel mapping is consistent with 11-limit Magic (41&22)[2], and also that of Miracle (31&41). Minerva ties in with the less important Magic variant called Telepathy (22&19e)[3] and also the mainstream extensions of Meantone (31&12) and Pajara (22&12). The union of Marvel and Minerva gives Orwell (31&22).

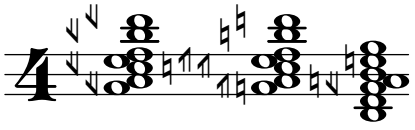
Then there's Apollo. It's consistent with 11-limit Magic (41&22) and Pajara (12&22), along with other notable temperaments. You can see it in Figure 3 and it looks nice and simple. It makes a compact shape on the lattice, covers the full 11-limit with five chains of fifths, and should be easy to write with a notation designed for 5-limit harmony. Unfortunately, tripod notation isn't designed for 5-limit harmony, and it works better with a slant, so it doesn't work so well with Apollo. You can use commas to get more 5-limit harmony and you need them for Apollo chords:



Another mapping that may deserve a mention (if not, I mentioned it anyway) is Prodigy. It's more complicated than 11-limit Marvel and doesn't give such an accurate temperament. Still, it's compatible with the mainstream 11-limit extensions of Miracle (31&41) and Meantone (31&12), so knowing the spelling might be useful. Here are some examples.



Another option is the beautifully simple but wildly inaccurate Potassium.

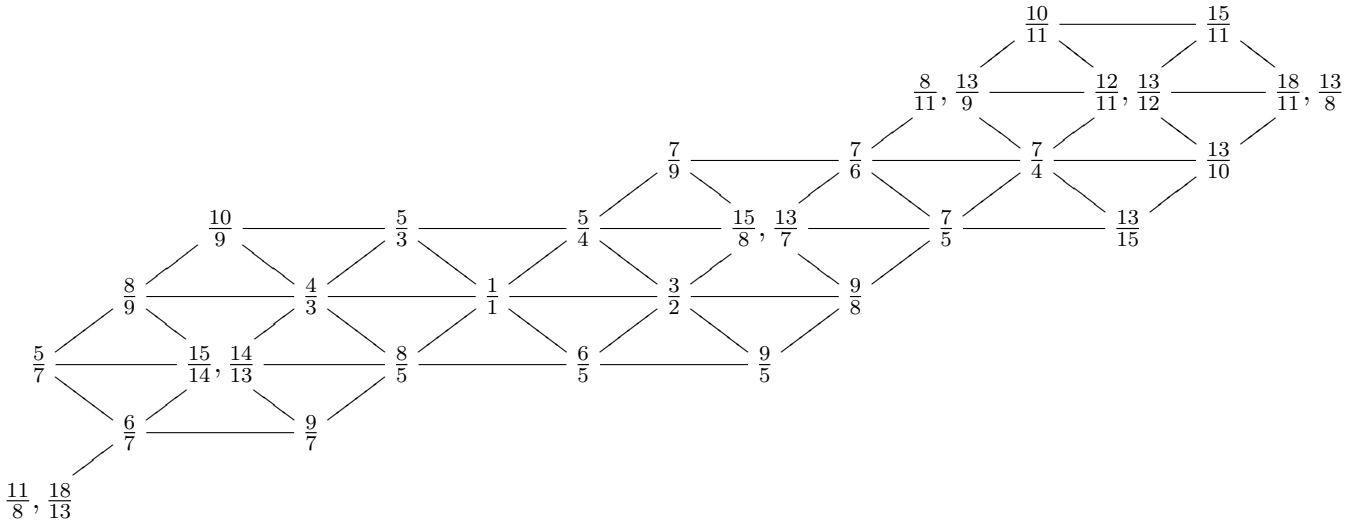


You may find a use for that if you think that a regular toe, approximating 16:15 or 15:14,

[2]The a&b notation gives a rank 2 temperament with a and b note equal tempered special cases with the obvious mappings. It's a way of uniquely specifying a rank 2 temperament class.

[3]The "e" means that the mapping of 19-equal is ambiguous, and the best mapping of all primes is chosen except for 11. The letters count prime numbers through the alphabet, with 2 as "a". This naming scheme was proposed by Herman Miller. I use a "p" to show the best approximation to all primes for an ambiguous equal temperament.

Figure 4: 15-limit ratios on the Tripod tempered lattice



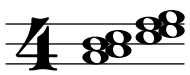
might be confused with a 12:11 that would normally be a neutral second. Given that, Potassium is outstanding, not only compared to other Marvel extensions (you may think Marvel is better suited to lower tuning errors) but next to all equivalently fuzzy rank 3 temperaments. It works with 11-limit Negri (9&10p), Meanenneadecal (12&7d), Pajaric (2&12), and August (12&9). The corresponding Magic variant is less notable. I call it Charisma (19p&3de).

Thirteen-Limit Extensions

A conventional thirteenth chord includes all the notes from a diatonic scale. It's what you get by putting your arm on the white keys of the piano. Although their origins are different, you can think of a 13-limit chord as being a retuned version of a thirteenth chord. Hopefully, the retuning will make it sound better. The intervals of 13 give you similar neutral intervals to the intervals of 11. That means that moving from 11 to 13 doesn't give you radically new sonorities the same way that moving from 9 to 11 does. But if you can use 11, why not 13? You could use 13 instead of 11, but here I'll go through the mappings that include both.

Marvels

You can introduce 13 into tripod notation by saying that the big toe that 12:11 maps to in Marvel is also a mapping of 13:12. As a consequence, you can divide 7:6 into unequal parts to get 12:13:14, and 14:13 maps the same as 15:14 and 16:15. Two versions of 12:13:14:15 can be written without any accidentals.

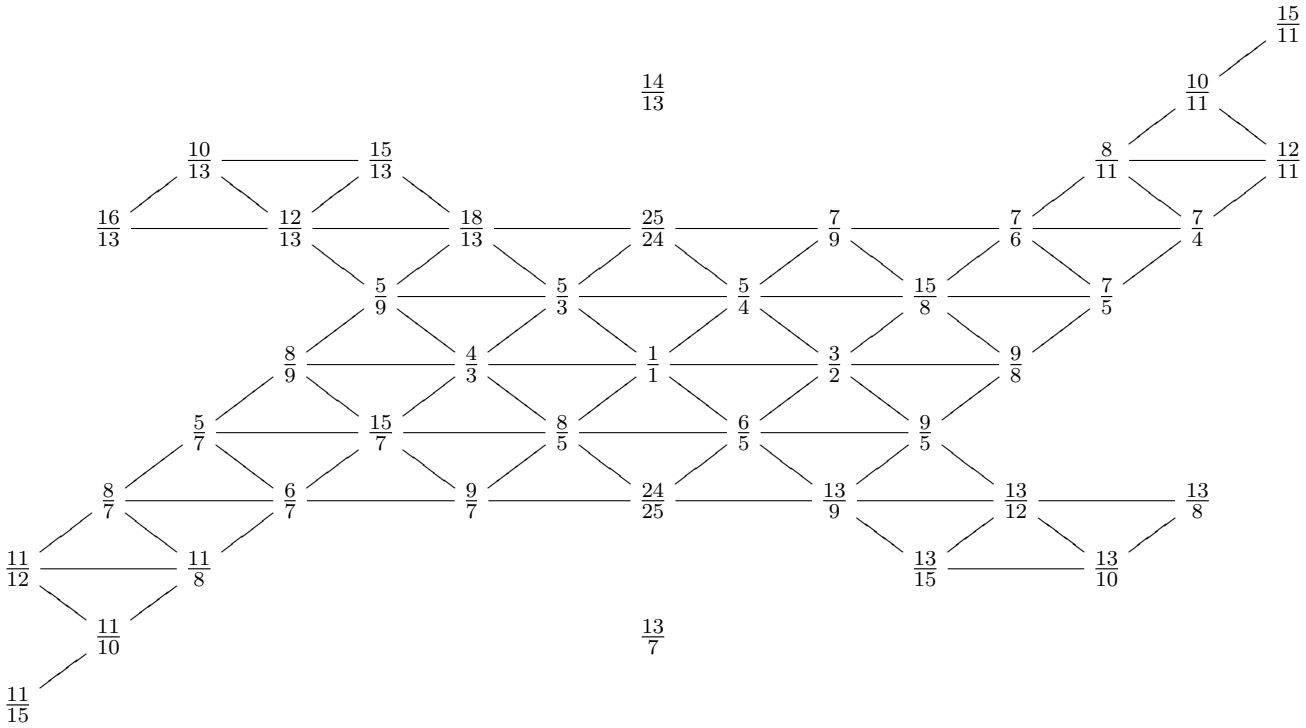


Because this fits tripod notation so well, I called its mapping Tripod. You can see its lattice in Figure 4. Most of the intervals of 13 duplicate other 11-limit intervals, but 13:10 is new.

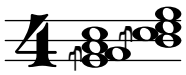
While I was working on this, Margo Schulter stated that "a 12:13:14:16 division is very

[4] M. Schulter, post to tuning@yahoogroups.com, Sun, 16 Oct 2011 22:52:12 -0700 (PDT).

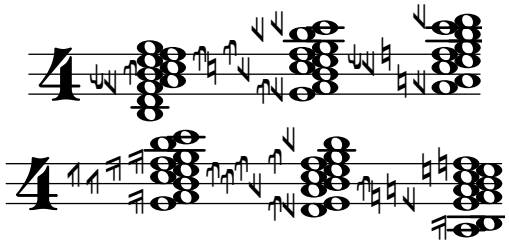
Figure 5: 15-limit ratios on the Hecate lattice



pleasing for a low variety of Persian Shur or Turkish Ushshaq."[4] You can write that in Tripod, but it needs an inch shift for the 13.

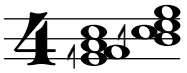


We may need inch shifts for general 13-limit chords, but because tripod notation gives us 9 staff positions to the octave, we can at least cram in a full 15-limit otonality (8:9:10:11:12:13:14:15:16). Here are some examples:

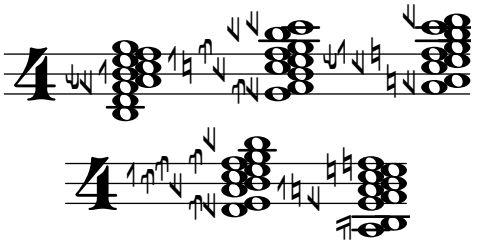


Tripod temperament works with 13-limit Magic (19p&41), the Orwell variant called Winston (9&31), Miraculous (31&41, a Miracle variant), and 13-limit Negri (9&19p).

Given that we need an inch shift to write the Shur/Ushshaq 12:13:14:16 tetrachord, why not change the inch to a comma? Doing so gets us closer to the ideal 13-limit intervals, and the system is called Hecate after the patron goddess of tripod notation.

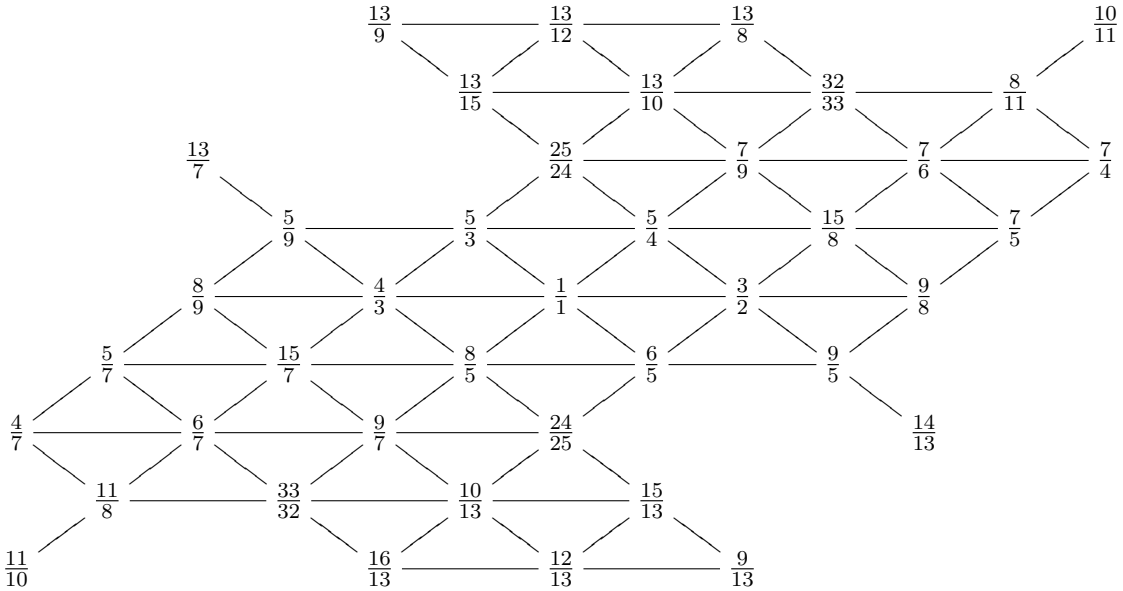


Hecate isn't as simple overall as Tripod. Here are some 15-limit otonalities.



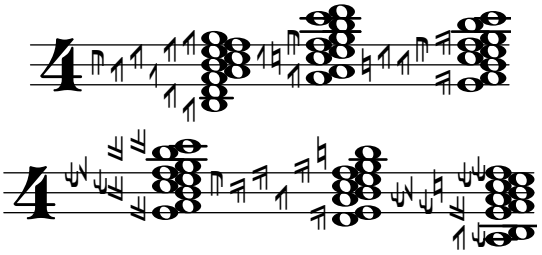
To see how Hecate maps to the lattice, take a look at figure 5. Every 15-limit ratio maps to a distinct pitch (except for 16:15 and 15:14, which get confused in the whole Marvel family, and so aren't both shown on any of the lattices) but the 15-limit points aren't connected. 14:13 and 13:7 are floating away from the main body, even with 25:24 added to the lattice.

Figure 6: 15-limit ratios on the Marvel lattice



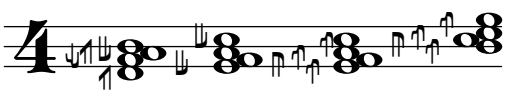
Hecate's followers include 13-limit Magic (19p&41), the Miracle variant called Manna (41&72), Catakleismic (72&53), and Garibaldi (41&53).

The mainstream 13-limit Marvel is comparable to Hecate: a little simpler, and a little less accurate. It works with 13-limit Orwell (31&53), Catakleismic, Benediction (31&72, a Miracle variant), Meanpop (31&19p), the lesser 13-limit meantone, and Sorcery (19p&22p), a simple Magic variant. You can get it by lowering the 13 of Tripod by a didymic comma. That makes it hard to find within the range of Tripod accidentals I've defined, but I managed to trap some.

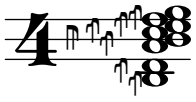


You can see 15-limit ratios mapped to the lattice according to Marvel temperament in Figure 6. It's a unique and efficient mapping in that, except for 16:15 and 15:14, no lattice points are mapped to by more than one 15-limit ratio and each 15-limit ratio is a neighbor of another 15-limit ratio.

Another reasonably accurate mapping is Isis. It isn't as accurate is Marvel or Hecate, and isn't appreciably simpler, but it's an alternative that might come in handy. Its followers include Miraculous (31&41), 13-limit Orwell (31&22p), the accurate Magic variant called Necromancy (41&22p), and Garibaldi (41&53). The 12:13:14:16 Shur/Ushaq tetrachord can be written as follows.



I can't work out a full 11-limit otonality, so here's an 8:9:11:12:13:14:15:16 chord.



Miraculous (31&41), 13-limit Orwell (31&22p), Necromancy (41&22p), and Garibaldi (41&51) are all notable followers of Isis.

A simple Marvel extension, called Mercury, is the complement of Tripod. You can write the 12:13:14:16 Shur/Ushaq tetrachord without accidentals.

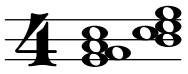
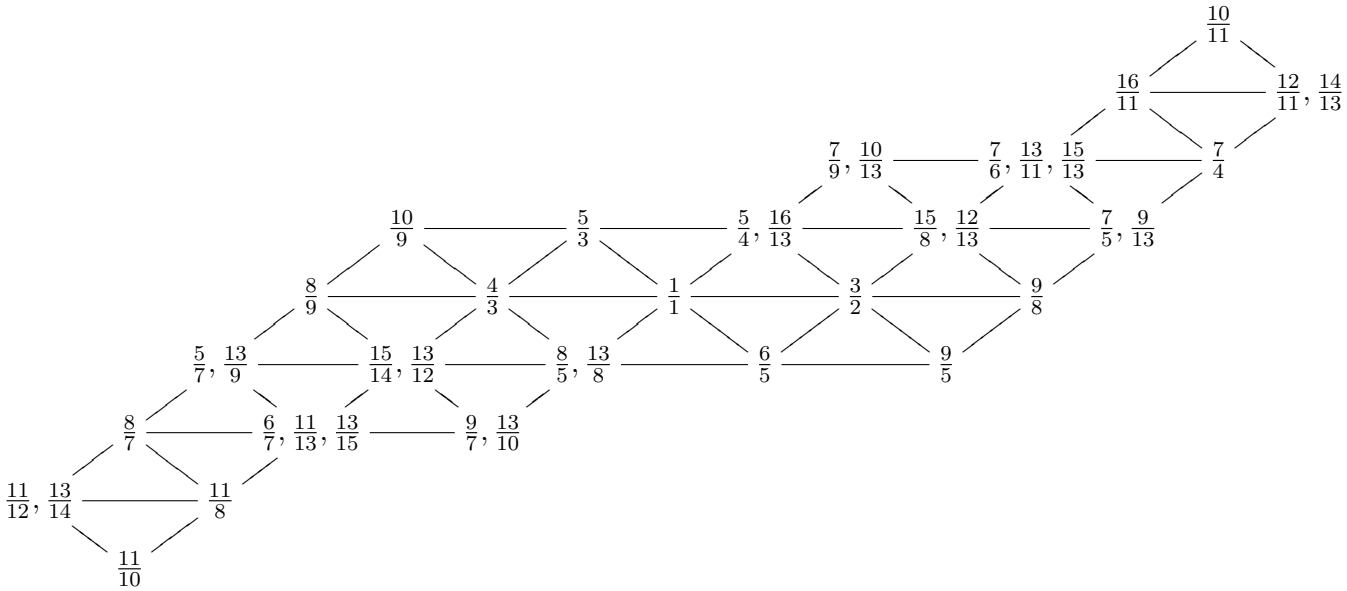
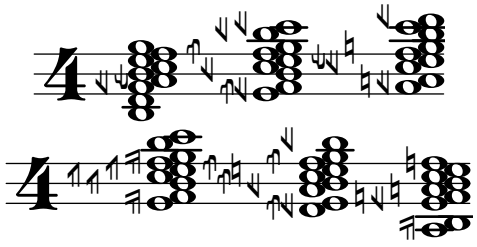


Figure 7: 15-limit ratios on the Mercury lattice



This defines 13:12 to be a smaller interval than 14:13, but 13:12 is obviously the larger ratio. As a result, Mercury doesn't get as close to the ideal 13-limit intervals as any of the other Marvel extensions I went through above. It's still worth looking at because it's very simple. Its followers include Negri (9&10p), the Orwell variant I call Blair (9&22p), and Sorcery (19p&22p). Here are some 15-limit otonalities.



There are different ways of approaching Mercury. The obvious one is with a great deal of disdain, because the 13:8 it approximates gets a long way from the 13:8 of acoustics. You may, however, feel that it gets close enough to give a 13-limit feel and the precise tuning doesn't matter so much. (The average error is close to that of 12 tone equal temperament for 5-limit triads. However, the 13-limit is much harder to make sense of than the 5-limit, so Mercury ends up sitting in the corner of the chamber reserved for apparently incomprehensible temperament classes. The one where the microphone doesn't reach, but nobody cares because they wouldn't understand anything even if they could hear it clearly.) You could try a tempered timbre to improve its concordance. You can even think about the tempered otonality as the Mercury chord, with no greater claim to acoustic privilege than the conventional thirteenth chords. When you do that, you can note that, as shown in the lattice in Figure 7, every interval of 13 in Mercury duplicates some other 15-limit interval. That makes the Mercury chord a tempered 15-limit chord regardless of 13 or its tuning.

The particular virtue of Mercury that makes it a notable spelling for tripod notation is that some chords that need an inch shift in Tripod lose it in Mercury. Say you're working with a keyboard that doesn't give you all the inches you'd like. You can pick out some ideas that sound reasonable with Mercury chords, and convert them to tripod notation. Then, you can try adding inch shifts to get Tripod spellings, or comma shifts to get Hecate spellings. With a mechanical rendering, you can listen to these theoretically purer tunings, and see if you prefer them. If you do, you managed to write something that wasn't possible on your keyboard, and that has to be good.

Figure 8: 15-limit ratios on the Minerva lattice

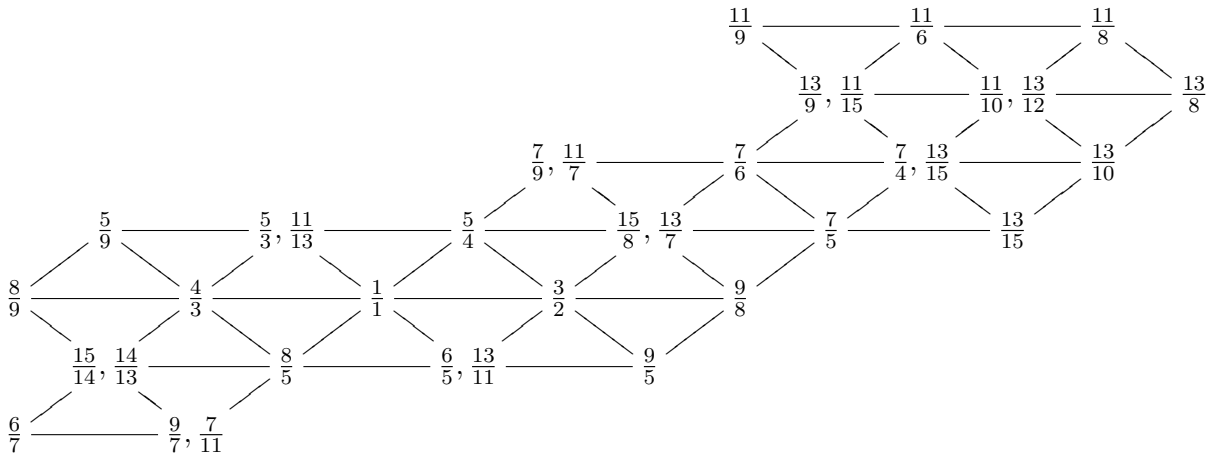
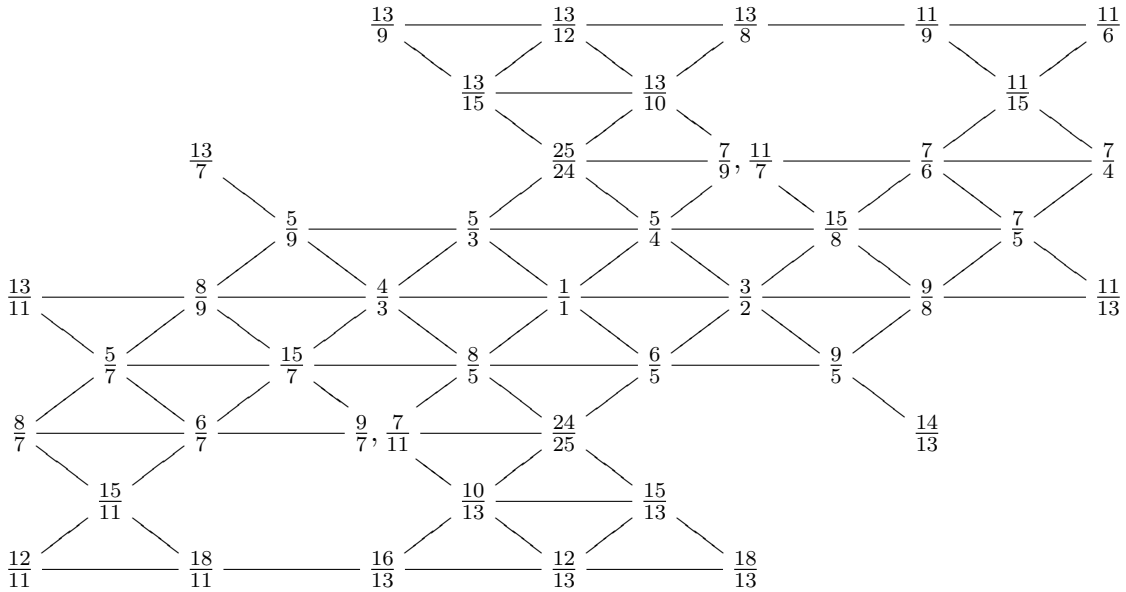
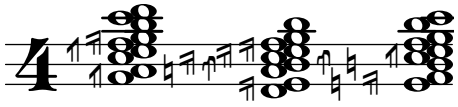


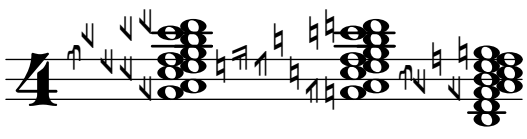
Figure 9: 15-limit ratios on the Athene lattice



Minervas

The mainstream extension of Minerva uses the same mapping of 13 as Tripod. It comes out simpler than Mercury, but barely more accurate. You can see it on the lattice in Figure 8. Like Tripod, the only interval of 13 that's distinct from other 15-limit intervals is 13:10. We can use it to write plenty of 15-limit otonalities.



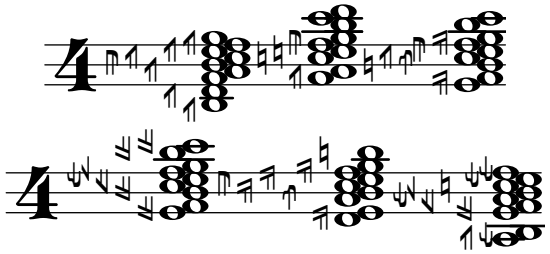


Minerva's most notable followers are August (9&12f), Negric (9&19e, a Negri variant), an anonymous Pajara extension (12f&10e), 13-limit Meantone (12f&19e), and Winston (9&22f). In the Magic family, it's represented by the fuzzy Telepathy extension called Intuition (19e&3def).

		2	3	5	7						2	3	5	7	
49:48	→ T|	−4	−1	0	2 ⟩ = |0	0	1 ⟩	81:80	→ T|	−4	4	−1	0 ⟩ = |	−1	2	1 ⟩
36:35	→ T|	2	2	−1	−1 ⟩ = |0	1	0 ⟩	25:24	→ T|	−3	−1	2	0 ⟩ = |	1	−1	0 ⟩
28:27	→ T|	2	−3	0	1 ⟩ = |1	−1	0 ⟩	50:49	→ T|	1	0	2	−2 ⟩ = |	1	−1	−1 ⟩
128:125	→ T|	7	0	−3	0 ⟩ = |0	1	0 ⟩	21:20	→ T|	−2	1	−1	1 ⟩ = |	0	1	1 ⟩
16:15	→ T|	4	−1	−1	0 ⟩ = |1	0	0 ⟩	15:8	→ T|	−3	1	1	0 ⟩ = |	8	1	3 ⟩
15:14	→ T|	−1	1	1	−1 ⟩ = |1	0	0 ⟩	15:7	→ T|	0	1	1	−1 ⟩ = |	10	1	3 ⟩
10:9	→ T|	1	−2	1	0 ⟩ = |2	−1	0 ⟩	9:5	→ T|	0	2	−1	0 ⟩ = |	7	2	3 ⟩
9:8	→ T|	−3	2	0	0 ⟩ = |1	1	1 ⟩	9:4	→ T|	−2	2	0	0 ⟩ = |	10	2	4 ⟩
8:7	→ T|	3	0	0	−1 ⟩ = |2	0	0 ⟩	7:4	→ T|	−2	0	0	1 ⟩ = |	7	1	3 ⟩
7:6	→ T|	−1	−1	0	1 ⟩ = |2	0	1 ⟩	12:7	→ T|	2	1	0	−1 ⟩ = |	7	1	2 ⟩
6:5	→ T|	1	1	−1	0 ⟩ = |2	1	1 ⟩	5:3	→ T|	0	−1	1	0 ⟩ = |	7	0	2 ⟩
5:4	→ T|	−2	0	1	0 ⟩ = |3	0	1 ⟩	8:5	→ T|	3	0	−1	0 ⟩ = |	6	1	2 ⟩
9:7	→ T|	0	2	0	−1 ⟩ = |3	1	1 ⟩	14:9	→ T|	1	−2	0	1 ⟩ = |	6	0	2 ⟩
4:3	→ T|	2	−1	0	0 ⟩ = |4	0	1 ⟩	3:2	→ T|	−1	1	0	0 ⟩ = |	5	1	2 ⟩
7:5	→ T|	0	0	−1	1 ⟩ = |4	1	2 ⟩	10:7	→ T|	1	0	1	−1 ⟩ = |	5	0	1 ⟩
2:1	→ T|	1	0	0	0 ⟩ = |9	1	3 ⟩									

Table 1: Some intervals mapped to the toe/semitoe/inch basis for Marvel.

A more accurate alternative is Athene. It extends 11-limit Minerva, but maps 13 the same way as 13-limit Marvel. It can get closer to the ideal 13-limit intervals than Tripod. Here are some 15-limit otonalities.



For Athene on the lattice, see Figure 9. Most 15-limit intervals map to distinct lattice points, and they form a connected region, after a fashion. The Athene mapping works with the mainstream 13-limit extensions of Meantone (12f&19e), Pajara (12f&22p), and Orwell (31&22p). The relevant member of the Magic family is 13-limit Telepathy (19e&22p).

Algebraic Definitions

Temperament mappings can be defined using matrix algebra. The following matrix converts ratio-space vectors into a description based on tripod notation.

$$\left|\begin{matrix}\langle\ 9 & 14 & 21 & 25|\\ \langle 19 & 30 & 44 & 53|\\ \langle\ 3 & 5 & 7 & 9|\end{matrix}\right\rangle$$

The first generator, described by the top row, is around 18 cents. It counts the number of steps on the tripod staff an interval is written with. The second generator, described by the middle row, is around 49 cents. It counts the number of semitoes (or steps on the 19 note Pengcheng scale) an interval approximates to. The last generator, described by the bottom row, is around 36 cents. It counts the number of feet an interval moves up on the staff.

I write the matrices in bra/ket form because that's the way some of us write temperament mappings. Most of the time I keep them in the correct orientation, so that you can ignore the bras and kets and follow the usual rules of matrix multiplication. I may change the occasional ket vector from col-

umn to row form to save space.

As an example, take the interval 16:15, written in vector form as $\left| \begin{array}{cccc} 4 & -1 & -1 & 0 \end{array} \right\rangle$. Multiply it by the Tripod matrix to see how to write it.

$$\begin{vmatrix} \langle & 9 & 14 & 21 & 25| \\ \langle & 19 & 30 & 44 & 53| \\ \langle & 3 & 5 & 7 & 9| \end{vmatrix} \Bigg\rangle \left| \begin{array}{c} 4 \\ -1 \\ 1 \\ 0 \end{array} \right\rangle$$

$$= \left| \begin{array}{c} 9 \times 4 + 14 \times -1 + 21 \times 1 + 25 \times 0 \\ 19 \times 4 + 20 \times -1 + 44 \times 1 + 53 \times 0 \\ 3 \times 4 + 5 \times -1 + 7 \times 1 + 9 \times 0 \end{array} \right\rangle$$

$$= \left| \begin{array}{ccc} 1 & 2 & 0 \end{array} \right\rangle$$

That tells you that 16:15 is a single step on the staff, two semitoes, and sits on a single foot. That makes it a toe.

That's fine for small intervals, but you may not want to count the semitoes for an interval that covers several staff positions. An alternative matrix, then, only counts semitoes relative to the staff position described by the first row. Let's call it "T" for Tripod.[5]

$$\mathrm{T} = \begin{vmatrix} \langle 9 & 14 & 21 & 25| \\ \langle 1 & 2 & 2 & 3| \\ \langle 3 & 5 & 7 & 9| \end{vmatrix} \Bigg\rangle \qquad (1)$$

A toe is then written as

$$\mathrm{T} \left| \begin{array}{cccc} 4 & -1 & 1 & 0 \end{array} \right\rangle = \left| \begin{array}{ccc} 1 & 0 & 0 \end{array} \right\rangle$$

To help you get oriented, there are some more intervals defined using this basis in Table 1. Most are intervals you might want to use in harmony. At the top, there are some small intervals that help you understand what Tripod accidentals mean.

As I said before, 16:15 is a single toe, as is 15:14. The two are equivalent in any Marvel temperament. The notated semitoe (for planar spellings) corresponds to an interval of 36:35. The rule that semitoes are measured relative to notated scale steps is broken for intervals of an octave or more because crossing the river gives you another semitoe.

The third component of tripod notation is the inch that gets added as you move from one foot to another. You can see that it approximates the ratio 49:48.

There are other intervals that approximate to semitoes or inches in Magic notation, and that can be written with alternative accidentals. The alternative semitoe is the difference between a toe and the primary semitoe. It approximates 25:24 and so can share a symbol with the usual sharp from conventional notation assuming a meantone temperament. The syntonic or didymic comma of 81:80 isn't such a simple interval in terms of the primary Tripod staff, but it is a kind of inch by Magic equivalences.

All of this is giving an algebraic formalism to the information that's already in the tripod notation examples and lattices. You may think that matrices are dry, scientific objects, but staff notation evokes the artistic nature of a talented musician.[6] However you feel, it makes no difference. The structure is the same however you write it.

Eleven-Limit Extensions

We will have a look at the 11-limit in a minute, honest. First, though, lets have another look at the 7-limit. With the CGI scripts on my website, if you look up 7-limit Marvel,[7] you'll find it gets listed with a "Reduced Mapping" that you may recognize

[5] For those who know their matrix multiplication, note that matrix 1 (T) is matrix multiplied on the left by

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Alternatively, you can subtract twice the top row from the middle row.

[6] You may think that notation is a contrivance that true musicians can do without. In which case, congratulations on getting this far.

[7] http://x31eq.com/cgi-bin/rt.cgi?ets=41_31_19&limit=7

	2	3	5	7	11		Marvel			Minerva			Prodigy			Apollo		
45:44 →	−2	2	1	0	−1	→	0	0	1	0	1	0	1	−1	−1	−1	2	1
33:32 →	−5	1	0	0	1	→	0	1	0	0	0	1	−1	2	2	1	−1	0
22:21 →	1	−1	0	−1	1	→	1	0	−1	1	−1	0	0	1	1	2	−2	−1
12:11 →	2	1	0	0	−1	→	1	0	1	1	1	0	2	−1	−1	0	2	1
11:10 →	−1	0	−1	0	1	→	1	1	0	1	0	1	0	2	2	2	−1	0
11:9 →	0	−2	0	0	1	→	3	0	0	3	−1	1	2	1	2	4	−2	0
14:11 →	1	0	0	1	−1	→	3	0	2	3	1	1	4	−1	0	2	2	2
11:8 →	−3	0	0	0	1	→	4	1	1	4	0	2	3	2	3	5	−1	1
16:11 →	4	0	0	0	−1	→	5	0	2	5	1	1	6	−1	0	4	2	2
11:7 →	0	0	0	−1	1	→	6	1	1	6	0	2	5	2	3	7	−1	1
18:11 →	1	2	0	0	−1	→	6	1	3	6	2	2	7	0	1	5	3	3
11:6 →	−1	−1	0	0	1	→	8	1	2	8	0	3	7	2	4	9	−1	2
11:5 →	0	0	−1	0	1	→	10	2	3	10	1	4	9	3	5	11	0	3

Table 2: Some 11-limit intervals mapped to the toe/semitoe/inch basis.

as Hermite normal form.[8] It looks like this:

$$\left|\begin{matrix}\langle 1 & 0 & 0 & -5| \\ \langle 0 & 1 & 0 & 2| \\ \langle 0 & 0 & 1 & 2|\end{matrix}\right\rangle$$

Because the left hand square part matches the identity matrix, you can turn it into the Tripod definition T (matrix 1) by multiplying it by the 5-limit part of T, which we can call T_5

$$T_5 = \left|\begin{matrix}\langle 9 & 4 & 21| \\ \langle 1 & 2 & 2| \\ \langle 3 & 5 & 7|\end{matrix}\right\rangle$$

That's useful for telling us something we already know.

$$\begin{aligned} & T_5 \left|\begin{matrix}\langle 1 & 0 & 0 & -5| \\ \langle 0 & 1 & 0 & 2| \\ \langle 0 & 0 & 1 & 2|\end{matrix}\right\rangle \\ = \; & T_5 \left\langle \left|\begin{matrix}1 \\ 0 \\ 0\end{matrix}\right\rangle \left|\begin{matrix}0 \\ 1 \\ 0\end{matrix}\right\rangle \left|\begin{matrix}0 \\ 0 \\ 1\end{matrix}\right\rangle \left|\begin{matrix}-5 \\ 2 \\ 2\end{matrix}\right\rangle \right| \\ = \; & \left|\begin{matrix}\langle 9 & 14 & 21 & 25| \\ \langle 1 & 2 & 2 & 3| \\ \langle 3 & 5 & 7 & 9|\end{matrix}\right\rangle = T \end{aligned}$$

And, yes, that is useful because we can also apply it to a higher limit mapping. Let's start with 11-limit Marvel[9].

$$\begin{aligned} M \; = \; & T_5 \left|\begin{matrix}\langle 1 & 0 & 0 & -5 & 12| \\ \langle 0 & 1 & 0 & 2 & -1| \\ \langle 0 & 0 & 1 & 2 & -3|\end{matrix}\right\rangle \\ = \; & \left|\begin{matrix}\langle 9 & 14 & 21 & 25 & 31| \\ \langle 1 & 2 & 2 & 3 & 4| \\ \langle 3 & 5 & 7 & 9 & 10|\end{matrix}\right\rangle \end{aligned}$$

This is the mapping for Minerva.[10]

$$\begin{aligned} N \; = \; & T_5 \left|\begin{matrix}\langle 1 & 0 & 0 & -5 & -9| \\ \langle 0 & 1 & 0 & 2 & 2| \\ \langle 0 & 0 & 1 & 2 & 4|\end{matrix}\right\rangle \\ = \; & \left|\begin{matrix}\langle 9 & 14 & 21 & 25 & 31| \\ \langle 1 & 2 & 2 & 3 & 3| \\ \langle 3 & 5 & 7 & 9 & 11|\end{matrix}\right\rangle \end{aligned}$$

If you want to turn a Minerva chord into a Marvel one, you need to know the comma that represents the difference between an approximate 11:8 in the two different mappings. That is

$$\left| \begin{matrix} -3 & 0 & 0 & 0 & 1 \end{matrix} \right\rangle (M - N) = \left| \begin{matrix} 0 & 1 & -1 \end{matrix} \right\rangle$$

It leads to a semitoe in Minerva becoming a comma in Marvel.

[8] Different references don't always agree on what Hermite normal form is. I found it easier to implement what I thought a lattice normal form should be, and have Gene Smith tell me it was Hermite. You could infer from the examples how they're being reduced, or you could have a look at Wikipedia.

[9] http://x31eq.com/cgi-bin/rt.cgi?ets=22_31_41&limit=11

[10] http://x31eq.com/cgi-bin/rt.cgi?ets=12_22_31&limit=11

Here's Apollo.[11]

$$\text{L} = \text{T}_5 \left\langle \begin{vmatrix} 1 & 0 & 0 & -5 & 2 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 2 & 2 \end{vmatrix} \right\rangle$$
$$= \left\langle \begin{vmatrix} 9 & 14 & 21 & 25 & 32 \\ 1 & 2 & 2 & 3 & 2 \\ 3 & 5 & 7 & 9 & 10 \end{vmatrix} \right\rangle$$

Prodigy[12].

$$\text{P} = \text{T}_5 \left\langle \begin{vmatrix} 1 & 0 & 0 & -5 & -13 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 2 & 3 \end{vmatrix} \right\rangle$$
$$= \left\langle \begin{vmatrix} 9 & 14 & 21 & 25 & 30 \\ 1 & 2 & 2 & 3 & 5 \\ 3 & 5 & 7 & 9 & 12 \end{vmatrix} \right\rangle$$

The simplest of all 11-limit mappings is Potassium.[13]

$$\text{K} = \text{T}_5 \left\langle \begin{vmatrix} 1 & 0 & 0 & -5 & -2 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{vmatrix} \right\rangle$$
$$= \left\langle \begin{vmatrix} 9 & 14 & 21 & 25 & 31 \\ 1 & 2 & 2 & 3 & 4 \\ 3 & 5 & 7 & 9 & 11 \end{vmatrix} \right\rangle$$

Finally, let's mention Artemis[14]. It has around the same error and complexity as Apollo and belongs to the Marvel family, but it doesn't have the same 7-limit geometry as Marvel so it doesn't map directly to tripod notation. This is the Hermite reduced basis:

$$\left\langle \begin{vmatrix} 1 & 0 & 1 & -2 & 2 \\ 0 & 1 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 & 4 \end{vmatrix} \right\rangle$$

To get it to work, you have to replace semitoes with quartertoes and inches with half-inches. (Scale degrees stay the same.) This is what you end up with:

$$\left\langle \begin{vmatrix} 9 & 14 & 21 & 25 & 31 \\ 2 & 4 & 4 & 6 & 7 \\ 6 & 10 & 14 & 18 & 21 \end{vmatrix} \right\rangle$$

It's a compromise between Marvel and Minerva, with intervals divided to get you midway between the two optimizations. Because it requires new alteration symbols, it isn't simple to write in tripod notation, and it's less accurate than Marvel. So it doesn't look very useful. You can easily find some new symbols from the sagittal set, though, and it does become more notable in higher limits, though. Here's its 19-limit extension:[15]

$$\left\langle \begin{vmatrix} 9 & 14 & 21 & 25 & 31 & 33 & 37 & 38 \\ 2 & 4 & 4 & 6 & 7 & 8 & 7 & 9 \\ 6 & 10 & 14 & 18 & 21 & 24 & 25 & 27 \end{vmatrix} \right\rangle$$

Whether that's useful or interesting to you depends on how you feel about the 19-limit. Some people do choose chords from it, and the extra precision you get by splitting accidentals makes your intentions clearer.

Thirteen-Limit Extensions

Here are the matrices and tables for some 13-limit temperaments, most of them already mentioned. Where two mappings extend the same 11-limit core, only the right hand columns are different. Where different temperaments are extended with the same mapping of 13, the right hand columns will match. From the tables, you can see other matching columns between different temperament classes that betray similarities in the way they get written.

13-limit Marvel[16].

$$\text{M} = \text{T}_5 \left\langle \begin{vmatrix} 1 & 0 & 0 & -5 & 12 & -4 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 & -3 & 4 \end{vmatrix} \right\rangle$$

[11] http://x31eq.com/cgi-bin/rt.cgi?ets=12_22_7d&limit=11
[12] http://x31eq.com/cgi-bin/rt.cgi?ets=10p_12_19e&limit=11
[13] http://x31eq.com/cgi-bin/rt.cgi?ets=12_9_7d&limit=11
[14] http://x31eq.com/cgi-bin/rt.cgi?ets=9_22_7d&limit=11
[15] http://x31eq.com/cgi-bin/rt.cgi?ets=29g_31_9p&limit=19
[16] http://x31eq.com/cgi-bin/rt.cgi?ets=19p_31_53&limit=13

	2	3	5	7	11	13		Tripod			Marvel			Hecate		
40:39 →	3	−1	1	0	0	−1	→	1	−1	−1	0	1	0	2	−3	−1
27:26 →	−1	3	0	0	0	−1	→	0	1	0	−1	3	1	1	−1	0
26:25 →	1	0	−2	0	0	1	→	0	1	1	1	−1	0	−1	3	1
14:13 →	1	0	0	1	0	−1	→	1	0	0	0	2	1	2	−2	0
13:12 →	−2	−1	0	0	0	1	→	1	0	1	2	−2	0	0	2	1
13:11 →	0	0	0	0	−1	1	→	2	0	2	3	−2	1	1	2	2
13:10 →	−1	0	−1	0	0	1	→	3	1	2	4	−1	1	2	3	2
18:13 →	1	2	0	0	0	−1	→	4	1	1	3	3	2	5	−1	1
13:9 →	0	−2	0	0	0	1	→	5	0	2	6	−2	1	4	2	2
20:13 →	2	0	1	0	0	−1	→	6	0	1	5	2	2	7	−2	1
22:13 →	1	0	0	0	1	−1	→	7	1	1	6	3	2	8	−1	1
13:7 →	0	0	0	−1	0	1	→	8	1	3	9	−1	2	7	3	3
25:13 →	0	0	2	0	0	−1	→	9	0	2	8	2	3	10	−2	2
13:6 →	−1	−1	0	0	0	1	→	10	1	4	11	−1	3	9	3	4

Table 3: Some 13-limit intervals mapped to the toe/semitoe/inch basis.

	2	3	5	7	11	13		Tripod			Isis			Mercury		
40:39 →	3	−1	1	0	0	−1	→	1	−1	−1	0	0	1	1	−1	0
27:26 →	−1	3	0	0	0	−1	→	0	1	0	−1	2	2	0	1	1
26:25 →	1	0	−2	0	0	1	→	0	1	1	1	0	−1	0	1	0
14:13 →	1	0	0	1	0	−1	→	1	0	0	0	1	2	1	0	1
13:12 →	−2	−1	0	0	0	1	→	1	0	1	2	−1	−1	1	0	0
13:11 →	0	0	0	0	−1	1	→	2	0	2	3	−1	0	2	0	1
13:10 →	−1	0	−1	0	0	1	→	3	1	2	4	0	0	3	1	1
18:13 →	1	2	0	0	0	−1	→	4	1	1	3	2	3	4	1	2
13:9 →	0	−2	0	0	0	1	→	5	0	2	6	−1	0	5	0	1
20:13 →	2	0	1	0	0	−1	→	6	0	1	5	1	3	6	0	2
22:13 →	1	0	0	0	1	−1	→	7	1	1	6	2	3	7	1	2
13:7 →	0	0	0	−1	0	1	→	8	1	3	9	0	1	8	1	2
25:13 →	0	0	2	0	0	−1	→	9	0	2	8	1	4	9	0	3
13:6 →	−1	−1	0	0	0	1	→	10	1	4	11	0	2	10	1	3

Table 4: Some 13-limit intervals mapped to the toe/semitoe/inch basis.

$$= \begin{vmatrix} \langle 9 & 14 & 21 & 25 & 31 & 34 | \\ \langle 1 & 2 & 2 & 3 & 4 & 2 | \\ \langle 3 & 5 & 7 & 9 & 10 & 11 | \end{vmatrix} \rangle$$

Tripod[17].

$$\mathrm{T} = \mathrm{T}_5 \begin{vmatrix} \langle 1 & 0 & 0 & -5 & 12 & -8 | \\ \langle 0 & 1 & 0 & 2 & -1 & 3 | \\ \langle 0 & 0 & 1 & 2 & -3 & 3 | \end{vmatrix} \rangle$$

$$= \begin{vmatrix} \langle 9 & 14 & 21 & 25 & 31 & 33 | \\ \langle 1 & 2 & 2 & 3 & 4 & 4 | \\ \langle 3 & 5 & 7 & 9 & 10 & 12 | \end{vmatrix} \rangle$$

Hecate[18].

$$\mathrm{H} = \mathrm{T}_5 \begin{vmatrix} \langle 1 & 0 & 0 & -5 & 12 & 2 | \\ \langle 0 & 1 & 0 & 2 & -1 & 4 | \\ \langle 0 & 0 & 1 & 2 & -3 & -2 | \end{vmatrix} \rangle$$

[17] http://x31eq.com/cgi-bin/rt.cgi?ets=19p_31_41&limit=13

[18] http://x31eq.com/cgi-bin/rt.cgi?ets=19p_53_41&limit=13

	2	3	5	7	11	13		Minerva			Athene			Prodigy		
40:39 →	3	−1	1	0	0	−1	→	1	−1	−1	0	1	0	1	−1	−1
27:26 →	−1	3	0	0	0	−1	→	0	1	0	−1	3	1	0	1	0
26:25 →	1	0	−2	0	0	1	→	0	1	1	1	−1	0	0	1	1
14:13 →	1	0	0	1	0	−1	→	1	0	0	0	2	1	1	0	0
13:12 →	−2	−1	0	0	0	1	→	1	0	1	2	−2	0	1	0	1
13:11 →	0	0	0	0	−1	1	→	2	1	1	3	−1	0	3	−1	0
13:10 →	−1	0	−1	0	0	1	→	3	1	2	4	−1	1	3	1	2
18:13 →	1	2	0	0	0	−1	→	4	1	1	3	3	2	4	1	1
13:9 →	0	−2	0	0	0	1	→	5	0	2	6	−2	1	5	0	2
20:13 →	2	0	1	0	0	−1	→	6	0	1	5	2	2	6	0	1
22:13 →	1	0	0	0	1	−1	→	7	0	2	6	2	3	6	2	3
13:7 →	0	0	0	−1	0	1	→	8	1	3	9	−1	2	8	1	3
25:13 →	0	0	2	0	0	−1	→	9	0	2	8	2	3	9	0	2
13:6 →	−1	−1	0	0	0	1	→	10	1	4	11	−1	3	10	1	4

Table 5: Some 13-limit intervals mapped to the toe/semitoe/inch basis.

$$= \begin{vmatrix} \langle 9 & 14 & 21 & 25 & 31 & 32 | \\ \langle 1 & 2 & 2 & 3 & 4 & 6 | \\ \langle 3 & 5 & 7 & 9 & 10 & 12 | \end{vmatrix} \rangle$$

Isis[19].

$$\mathrm{S} = \mathrm{T}_5 \begin{vmatrix} \langle 1 & 0 & 0 & -5 & 12 & 17 | \\ \langle 0 & 1 & 0 & 2 & -1 & -4 | \\ \langle 0 & 0 & 1 & 2 & -3 & -3 | \end{vmatrix} \rangle$$

$$= \begin{vmatrix} \langle 9 & 14 & 21 & 25 & 31 & 34 | \\ \langle 1 & 2 & 2 & 3 & 4 & 3 | \\ \langle 3 & 5 & 7 & 9 & 10 & 10 | \end{vmatrix} \rangle$$

Mercury[20].

$$\mathrm{Hg} = \mathrm{T}_5 \begin{vmatrix} \langle 1 & 0 & 0 & -5 & 12 & 6 | \\ \langle 0 & 1 & 0 & 2 & -1 & 0 | \\ \langle 0 & 0 & 1 & 2 & -3 & -1 | \end{vmatrix} \rangle$$

$$= \begin{vmatrix} \langle 9 & 14 & 21 & 25 & 31 & 33 | \\ \langle 1 & 2 & 2 & 3 & 4 & 4 | \\ \langle 3 & 5 & 7 & 9 & 10 & 11 | \end{vmatrix} \rangle$$

Minerva[21].

$$\mathrm{N} = \mathrm{T}_5 \begin{vmatrix} \langle 1 & 0 & 0 & -5 & -9 & -8 | \\ \langle 0 & 1 & 0 & 2 & 2 & 3 | \\ \langle 0 & 0 & 1 & 2 & 4 & 3 | \end{vmatrix} \rangle$$

$$= \begin{vmatrix} \langle 9 & 14 & 21 & 25 & 31 & 33 | \\ \langle 1 & 2 & 2 & 3 & 3 & 4 | \\ \langle 3 & 5 & 7 & 9 & 11 & 12 | \end{vmatrix} \rangle$$

Athene[22].

$$\mathrm{N} = \mathrm{T}_5 \begin{vmatrix} \langle 1 & 0 & 0 & -5 & -9 & -4 | \\ \langle 0 & 1 & 0 & 2 & 2 & -1 | \\ \langle 0 & 0 & 1 & 2 & 4 & 4 | \end{vmatrix} \rangle$$

$$= \begin{vmatrix} \langle 9 & 14 & 21 & 25 & 31 & 34 | \\ \langle 1 & 2 & 2 & 3 & 3 & 2 | \\ \langle 3 & 5 & 7 & 9 & 11 & 11 | \end{vmatrix} \rangle$$

13-limit Prodigy[23] uses the same mapping of 13 as Tripod and Minerva.

$$\mathrm{P} = \mathrm{T}_5 \begin{vmatrix} \langle 1 & 0 & 0 & -5 & -13 & -8 | \\ \langle 0 & 1 & 0 & 2 & 6 & 3 | \\ \langle 0 & 0 & 1 & 2 & 3 & 3 | \end{vmatrix} \rangle$$

$$= \begin{vmatrix} \langle 9 & 14 & 21 & 25 & 30 & 33 | \\ \langle 1 & 2 & 2 & 3 & 5 & 4 | \\ \langle 3 & 5 & 7 & 9 & 12 & 12 | \end{vmatrix} \rangle$$

[19] http://x31eq.com/cgi-bin/rt.cgi?ets=53_41_31&limit=13
[20] http://x31eq.com/cgi-bin/rt.cgi?ets=22p_9_19p&limit=13
[21] http://x31eq.com/cgi-bin/rt.cgi?ets=9_19e_31&limit=13
[22] http://x31eq.com/cgi-bin/rt.cgi?ets=53_19e_31&limit=13
[23] http://x31eq.com/cgi-bin/rt.cgi?ets=10p_12f_19e&limit=13

Change Log

November 2014 fixed 11-limit Prodigy to be an 8:9:10:11:12:14:16 chord. (The 14 was wrong before.) Also made the Potassium below 8:9:10:11:12:14:16 and tweaked margins to stay close to the original pagination.