RMS-Based Error and Complexity Measures Involving Composite Intervals

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This article follows on from *Primerr*.[1] There, I talked about errors and complexities as a function of the weighted errors or mappings of prime intervals. Here, I talk about the error as the root mean squared (RMS) deviation of a set of intervals from just intonation. For a regular temperament this RMS error is still independent of the number of notes in the tempered scale.

A model of composite intervals is a bit less arbitrary than one that only looks at prime intervals. In that sense the errors and complexities I talk about here are more meaningful than those in *Primerr*. Where the two are in agreement, you can view this article as validating *Primerr*. However, choosing a set of intervals that doesn't repeat every octave, and the weight to assign to each interval, still requires a lot of arbitrary choices. From that viewpoint, everything here is still wrong, but not as wrong as in *Primerr*.

I show the general formula for the weighted RMS error of a set of composite intervals in the optimal tuning of a given temperament class, where the tuning itself remains implicit. The formula is not much more complicated than that for the TOP-RMS error.

The same formula also leads to a measure of the complexity of a regular temperament. I don't derive it from first principles but it's probably something to do with the RMS complexity of the given intervals.

I discuss the special case of equally weighted errors within a given Tenney (product) limit and show that in all cases the formula is identical to that for the RMS error of prime intervals with arbitrary weights.

To cover more general cases, I propose a simple model for systems that don't only depend on the prime weights. In the light of this I look at Farey limits, where all intervals are taken from the lowest partials in a harmonic overtone series.

There's a good chance that I'm re-discovering well known 19th Century mathematics. That may appear wasteful, but if it isn't well known to me I'm not sure how to find out about it. Now I've written it up at least it saves you from working it out yourself.

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[1] "Prime Based Error and Complexity Measures", Available at `http://x31eq.com/primerr.pdf`. I'll call this *Primerr* throughout for the sake of brevity.

1 Measures of Composite Intervals

1.1 RMS Errors of Composite Intervals

In *Primerr* I gave formulas for a prime-based RMS error using an implicit weighting. Here, I'll change the notation and make the weighting explicit. The matrix M now represents the unweighted mapping and W is a matrix containing the prime weights where

$$w_{ii} = \frac{1}{b_i} \tag{1}$$

$$w_{ij} = 0, i \neq j \tag{2}$$

with b_i as the buoyancy for the ith prime. H is a column vector containing the sizes of the prime intervals we're trying to approximate. The approximation we want to solve can then be written as

$$WM\mathbf{g} \simeq WH \tag{3}$$

with $\mathbf{g}$ as the generators of the temperament.

The next step is to define a matrix C where each row defines a composite interval according to H. That is, the intervals we want to approximate are now CH. In these terms, the approximation becomes

$$WCM\mathbf{g} \simeq WCH \tag{4}$$

The weightings W now refer to these composite intervals, rather than the prime intervals.

That leads to a formula for the sum-square error similar to Equation 27 on p. 8 of *Primerr*

$$E^2 = (WCM\mathbf{g} - WCH)^T(WCM\mathbf{g} - WCH) \tag{5}$$

1.2 Optimal Errors

If you like, you can solve the problem for $\mathbf{g}$ that give the least squares optimum. I'm more interested in going straight to the optimal error.[2]

Theorem 1 *The optimal error E_{opt} of composite intervals C with weights W defined according to prime intervals H with a mapping from primes to generators of M is given by*

$$E_{\text{opt}}^2 = \frac{\left| M^T G M - \frac{M^T G H H^T G M}{H^T G H} \right|}{|M^T G M|} \tag{6}$$

where

$$G = C^T W^2 C \tag{7}$$

To see this, start with Equation 32 on p. 9 of *Primerr* and put the weights in explicitly.

$$E_{\text{opt}}^2 = \frac{\left| M^T W^2 M - \frac{M^T W^2 H H^T W^2 M}{H^T W^2 H} \right|}{|M^T W^2 M|} \tag{8}$$

Then, replace every M with CM and every H with CH.

$$E_{\text{opt}}^2 = \frac{\left| M^T C^T W^2 C M - \frac{M^T C^T W^2 C H H^T C^T W^2 C M}{H^T C^T W^2 C H} \right|}{|M^T C^T W^2 C M|} \tag{9}$$

In this equation, $C^T W^2 C$ keeps cropping up, so replace it with G and you get Theorem 1.

1.3 The Metric and Normalization

I call G a metric because it defines how to measure the size of the errors.[3] It has an interesting property in that it's always an $n \times n$ matrix for n prime intervals. That is, the basic problem is the same regardless of how many composite intervals you define. This is good because the number of intervals between harmonics tends to grow as the square of the number of harmonics you're interested in. And the number of harmonics can never be smaller than the number of prime intervals.

Because the metric can be smaller than the matrices that define it, there are different composite intervals that can give the same metric, given the right weighting. It's also likely that different problem definitions will give metrics that look very similar. However, if the weights aren't normalized correctly the numbers in the metric can get arbitrarily large. To make metrics more easily comparable, I'll define a normalized metric, which is simply defined as the metric divided by its top-left entry.

$$G' = \frac{G}{G_{00}} \tag{10}$$

That uses my wacky notation where indexes start at zero, so G_{00} is the entry that corresponds to octaves matching with octaves. With Tenney weighting of primes, G_{00} is always one, and this normalization defines it to be so for arbitrary sets of composites and weights.

Equation 6 is properly normalized in that if you multiply G by a constant the result is unaffected. However, Equation 5 is still proportional to the sizes of the weights. To normalize it, first write it in terms of G.

$$\begin{aligned} E^2 &= [WC(M\mathbf{g} - WCH)]^T(WCM\mathbf{g} - WCH) \\ &= (M\mathbf{g} - H)^T(WC)^T(WC)(M\mathbf{g} - H) \\ &= (M\mathbf{g} - H)^T G(M\mathbf{g} - H) \end{aligned} \tag{11}$$

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[2]This is strictly speaking the optimal, normalized, weighted, RMS error. But I don't want to say that every time I mention it, so I'll call it "the optimal error" instead.

[3]Properly speaking, the matrix G is something like a metric tensor rather than the metric itself. But I don't speak properly so I'll call it "the metric".

Then, divide through by the size of H in terms of the metric

$$E_{\text{norm}}^2 = \frac{(M\mathbf{g} - H)^T G(M\mathbf{g} - H)}{H^T G H} \qquad (12)$$

Here, E_{norm} is the normalized, RMS error whose optimum is given in Equation 6. $H^T G H$ is a real number, and a constant for a given choice of composite intervals and weights.

1.4 Scalar Complexity and Badness

The scalar complexity follows from Equation 76 on p. 18 of *Primerr* as

$$k^2 = \left| \frac{M^T G M}{H^T G H} \right| \qquad (13)$$

Interpreting G as a metric, that's the volume of M normalized by the length of H.

Scalar badness defined as error times complexity is

$$B^2 = \left| \frac{M^T G M}{H^T G H} - \frac{M^T G H H^T G M}{H^T G H H^T G H} \right| \qquad (14)$$

Later on, I'll suggest a different way of normalizing the error and, by implication, the badness.[4] But these formulas will do for now.

2 Equally Weighted Tenney Limits

The Tenney (or product) limit L can be defined (Erlich 2004) as all ratios $n : d$ in their simplest terms such that $n \times d \leq L$. Where $n : d$ represents a musical interval you can assume $0 < n \leq d$.[5] I'll use the shorthand T_L for the set of such intervals within the Tenney limit of L.

The idea of a Tenney limit is that an interval is more useful the simpler its ratio is. The product of n and d is a function of the Tenney harmonic distance. (Tenney 1984)

The Tenney limit on its own isn't usually what you want because it brings in too many high primes. For example, to get traditional harmony you need $5{:}4$. So you set a Tenney limit of 20. Then, you also have intervals like $19{:}1$, $17{:}1$, and so on. To correct for this, define a prime limit P_p containing the ratios that don't contain prime factors larger than p. Then, use the intersection of a Tenney and prime limit notated as $T_L \cap P_p$.

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[4]Generally speaking, badness is a function of error and complexity that gives you an idea of how bad a temperament or temperament class is. Today, though, I use it to mean scalar badness in particular, or error times complexity. I'm not interested in how bad the temperaments are.

[5]You can also specify the Tenney limit as the geometric mean (square root of the product) of n and d or as the logarithm of the product of n and d. For this article I use the product because it means each distinct Tenney limit can be specified with an integer.

You can happily use arbitrary weights with a Tenney limit. Today, though, lets use equal weighting so that $W = I$. Then, a Tenney limit L and a prime limit P leads to a unique metric $G(T_L \cap P_p)$.

2.1 Pythagorean Limits

The intersections of Tenny limits with the 3-limit can all be described as Pythagorean. Anything simpler would only have one prime interval, and so wouldn't support a regular temperament. The only regular temperament for a Pythagorean limit is an equal temperament. There aren't likely to be any new and exciting properties of Pythagorean equal temperaments waiting to be discovered. However, taking a detailed look at Pythagorean limits does help to illustrate the general behavior of the metrics before we move on to more complicated examples.

The simplest Pythagorean Tenney limit is T_3. It contains the intervals 1:1, 2:1 and 3:1. With equal weighting, the metric is

$$G(T_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So far, so exciting! Next up is the 4-limit, with the new ratio 4:1.

$$G(T_4) = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 2^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$$

The element relating to octaves is the sum of the squares of the number of factors of two in each ratio.

The Pythagorean 5-limit doesn't contain any ratios that weren't in the 4-limit. So let's go straight to the 6-limit. There are two new ratios with a product of 6, being 6:1

Table 1: Errors in cents for some Pythagorean equal temperaments with different Tenney (product) limits

Mapping	Pure octave RMS					Optimal RMS				
	12	72	144	432	864	12	72	144	432	864
$\langle 5,8|$	18.04	28.87	32.64	40.35	45.69	12.89	20.92	23.91	28.94	32.42
$\langle 7,11|$	16.24	25.99	29.38	36.32	41.12	11.70	18.99	21.70	26.27	29.44
$\langle 12,19|$	1.96	3.13	3.54	4.37	4.95	1.40	2.28	2.60	3.15	3.53
$\langle 17,27|$	3.93	6.28	7.10	8.78	9.94	2.82	4.57	5.22	6.32	7.08
$\langle 19,30|$	7.22	11.55	13.06	16.14	18.27	5.19	8.42	9.62	11.65	13.05
$\langle 22,35|$	7.14	11.42	12.91	15.96	18.07	5.11	8.29	9.48	11.47	12.86
$\langle 29,46|$	1.49	2.39	2.70	3.34	3.78	1.07	1.74	1.99	2.40	2.69
$\langle 31,49|$	5.18	8.29	9.37	11.58	13.12	3.72	6.04	6.90	8.36	9.36
$\langle 41,65|$	0.48	0.77	0.88	1.08	1.23	0.35	0.56	0.64	0.78	0.87
$\langle 53,84|$	0.07	0.11	0.12	0.15	0.17	0.05	0.08	0.09	0.11	0.12

Table 2: Figures for some Pythagorean equal temperaments with different Tenney (product) limits

Mapping	Optimal error (cent/oct)					Octave stretch (cent)				
	12	72	144	432	864	12	72	144	432	864
$\langle 5,8|$	5.664	5.657	5.649	5.663	5.665	-5.52	-5.36	-5.23	-5.48	-5.60
$\langle 7,11|$	5.142	5.135	5.126	5.141	5.144	4.97	4.82	4.70	4.93	5.04
$\langle 12,19|$	0.617	0.616	0.615	0.617	0.617	0.60	0.58	0.57	0.59	0.61
$\langle 17,27|$	1.237	1.236	1.234	1.237	1.238	-1.20	-1.17	-1.14	-1.19	-1.22
$\langle 19,30|$	2.280	2.277	2.273	2.280	2.281	2.21	2.14	2.09	2.19	2.24
$\langle 22,35|$	2.246	2.243	2.240	2.245	2.247	-2.18	-2.12	-2.07	-2.17	-2.21
$\langle 29,46|$	0.471	0.470	0.469	0.471	0.471	-0.46	-0.44	-0.43	-0.45	-0.46
$\langle 31,49|$	1.636	1.634	1.631	1.635	1.636	1.59	1.54	1.50	1.57	1.61
$\langle 41,65|$	0.153	0.152	0.152	0.153	0.153	-0.15	-0.14	-0.14	-0.15	-0.15
$\langle 53,84|$	0.022	0.021	0.021	0.022	0.022	0.02	0.02	0.02	0.02	0.02

and $3{:}2$.

$$G(T_6 \cap P_3) = \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+1+1+2^2 & -1+1 \\ -1+1 & 1+1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}$$

Again, the diagonal elements are the sums of the squares of the numbers of times each prime occurs as a factor in each ratio. Fortunately, the off-diagonals are still zero.

The next non-trivial limit is 8. That gives us the new

ratio $8{:}1$.

$$G(T_8 \cap P_3) = \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+1+1+2^2+3^2 & -1+1 \\ -1+1 & 1+1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 0 \\ 0 & 3 \end{pmatrix}$$

The 9-limit adds $9{:}1$.

$$T_9 \cap P_3 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 1 & 1 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}$$

$$G(T_9 \cap P_3) = \begin{pmatrix} 1+1+1+2^2+3^2 & -1+1 \\ -1+1 & 1+1+2^2+1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 0 \\ 0 & 7 \end{pmatrix}$$

The next distinct limit is 12. It gives us the new ratios $12{:}1$ and $4{:}3$

$$T_{12} \cap P_3 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 1 & 1 \\ 2 & -1 \\ 2 & 0 \\ 2 & 1 \\ 3 & 0 \end{pmatrix}$$

$$G(T_{12} \cap P_3) = \begin{pmatrix} 1 \times 3 + 2^2 \times 3 + 3^2 & -1+1-2+2 \\ -1+1-2+2 & 1 \times 5 + 2^2 \end{pmatrix}$$

$$= \begin{pmatrix} 24 & 0 \\ 0 & 9 \end{pmatrix}$$

This is the first limit that gives us full-blooded harmony because it allows you to write a 2:3:4 chord. Fortunately it can still be expressed as a prime weighting.

Maybe you've noticed the pattern with the distinct limits. If a new limit brings in a new interval, it must have no prime factors other than 2 and 3. So the next distinct limit will be 16, which brings in $16{:}1$ and nothing else. So not very exciting. After that there's 18. So you get $18{:}1$ and $9{:}2$. Well, I suppose $9{:}2$ is an interval you might use, but when does $9{:}8$ come into the picture? Right up in the 72-limit, so let's go straight there.

$$G(T_{72} \cap P_3) = \begin{pmatrix} 181 & 0 \\ 0 & 64 \end{pmatrix}$$

Well, hey, the off-diagonals are still zero! Could there be a pattern here? The answer is yes.

Theorem 2 *For any Pythagorean subset of a Tenney limit, the off-diagonal elements of the metric are always zero. Hence the unweighted RMS of the subset of the Tenney limit is the same as a weighted RMS of the prime limit.*

To prove this, first note that the two off-diagonal elements are always the same, and equal to the sum of the factors of 2 multiplied by the factors of 3. From this it's fairly obvious that prime powers won't contribute to those terms because they don't have any factors of either 2 or 3 and anything multiplied zero is zero. But let's call it a lemma anyway.

Lemma 1 *The off-diagonal elements of the metric for a Tenney limit are independent of intervals of the form $p^n{:}1$ where p is a prime number and n is a positive integer.*

Note that this is not restricted to Pythagorean limits. However many primes you have prime powers themselves only count towards the diagonals of the metric.

That tells us that the intervals we're interested in are all of the form $|a, b\rangle$ where a and b are both non-zero integers. The ket notation means that the ratio is of the form $2^a{:}3^b$. This leads to a new lemma.

Lemma 2 *Any interval with only two prime factors, p and q, with exponents a and b respectively, comes in two forms within a given Tenney limit: $p^a{:}q^b$ and $p^a q^b{:}1$ where p and q are chosen so that $p^a > q^b$.*

This is true because $p^a{:}q^b$ and $p^a q^b{:}1$ will always have the same product, and so lie within the same Tenney limits. Also, as p and q are primes, there are no other ratios in their lowest terms involving only p^a and q^b.

These lemmas tell as that all the intervals that contribute to the off-diagonals come in pairs of the form $|a, b\rangle$ and $|a, -b\rangle$ where a and b are integers. (They may not be the same integers as above, and may not both be positive, but that doesn't matter.) There total contribution to the metric is therefore $ab - ab$ which is always zero. Hence Theorem 2

If $9{:}8$ is useful in harmony, maybe $27{:}16$ is as well. That leads to

$$G(T_{432} \cap P_3) = \begin{pmatrix} 664 & 0 \\ 0 & 245 \end{pmatrix}$$

Perhaps you're a real devotee of Pythagorean harmony, and you want to ensure your $81{:}64$ is in tune as well. In that case, you need

$$G(T_{5184} \cap P_3) = \begin{pmatrix} 2524 & 0 \\ 0 & 992 \end{pmatrix}$$

The numbers are getting quite big, so how about the harmonically complete limits above in a normalized,

form, for more easy comparison?

$$G(T_{12} \cap P_3) = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 0.375 \end{pmatrix}$$

$$G(T_{72} \cap P_3) = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 0.354 \end{pmatrix}$$

$$G(T_{144} \cap P_3) = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 0.338 \end{pmatrix}$$

$$G(T_{432} \cap P_3) = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 0.369 \end{pmatrix}$$

$$G(T_{864} \cap P_3) = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 0.385 \end{pmatrix}$$

$$G(T_{5184} \cap P_3) = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 0.393 \end{pmatrix}$$

That list also includes T_{144} so that you can get 16:9 along with 9:8 and T_{864} so that you can get 32:27 along with 27:16. Under octave equivalence, the pairs of metrics would have to be merged somehow.

Those numbers are getting close to the Tenney weighting of

$$\frac{1}{[\log_2(3)]^2} = 0.398072\ldots \tag{15}$$

You can see that Tenney weighting should be the outcome of a Tenney limit extrapolated to infinity. The diagonal entries are the sums of squares of the number of factors of a given prime in each ratio. As ratios get arbitrarily large the number of times each prime gets used should be inversely proportional to the logarithm of its size, and so as large ratios dominate the total should be inversely proportional to the square of the logarithm. However, I can't prove this.

Table 1 shows the errors for different equal temperaments. The errors on the left hand side assume a tuning with pure octaves. On the right hand side, the tuning is chosen to optimize the RMS error. Naturally, the optimized errors are generally lower than those for pure octaves. Because there are only two primes involved it doesn't matter if you do the optimization or not when comparing temperaments as long as you always compare pure octave temperaments with other pure octave temperaments.

You can see that the errors get larger the higher the Tenney limit gets. This is the trend that normalization is designed to counteract.[6] You can some the normalized

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[6]Table 1 is supposed to show the raw RMS errors in cents. It may not include 1:1 in the average, though. This doesn't affect the relative errors but will change the absolute value, so be aware of this if you want to compare you figures with mine.

errors in Table 2 and, sure enough, they're more consistent. So consistent, in fact, that you may wonder what the point is of specifying the Tenney limit in the first place.

Table 2 also shows how much the octaves get stretched to give the optimal RMS error for each Tenney limit. These figures are fairly consistent as well.

2.2 5-Prime Limits

The simplest Tenney limit within the 5-prime limit is, of course, T_5. That contains intervals 1:1, 2:1, 3:1, 4:1, and 5:1. The metric for equal weighting is

$$G(T_5) = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Next is the 6-limit, with 6:1 and 3:2 added

$$G(T_6) = \begin{pmatrix} 0 & -1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 3 + 2^2 & 1-1 & 0 \\ 1-1 & 1 \times 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 8-limit only adds 8:1

$$T_8 \cap P_5 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

Table 3: Errors and the like for some 5-prime limit equal temperaments

Product limit	20	40	80	120	240	20	40	80	120	240
Mapping	Pure octave RMS (cent)					Optimal RMS (cent)				
$\langle 7, 11, 16|$	30.98	36.73	44.19	46.80	53.39	20.46	24.48	29.08	30.24	34.59
$\langle 12, 19, 28|$	8.53	10.22	12.53	13.03	14.99	7.90	9.38	11.22	11.90	13.56
$\langle 15, 24, 35|$	19.87	22.95	26.21	29.20	32.60	12.98	15.19	17.23	18.73	21.02
$\langle 19, 30, 44|$	8.49	9.86	11.38	12.55	14.07	5.36	6.30	7.19	7.73	8.71
$\langle 22, 35, 51|$	7.64	8.81	10.01	11.20	12.48	7.11	8.30	9.61	10.53	11.84
$\langle 31, 49, 72|$	5.20	5.96	6.69	7.58	8.40	4.39	5.10	5.84	6.44	7.21
$\langle 34, 54, 79|$	4.10	4.71	5.33	5.99	6.66	2.87	3.34	3.78	4.15	4.64
$\langle 41, 65, 95|$	3.57	4.28	5.26	5.46	6.28	3.21	3.82	4.57	4.83	5.51
$\langle 46, 73, 107|$	3.86	4.55	5.43	5.80	6.59	2.43	2.90	3.41	3.56	4.07
$\langle 53, 84, 123|$	0.86	1.03	1.27	1.31	1.51	0.71	0.84	1.01	1.06	1.21
Mapping	Normalized optimum (cent/oct)					Octave stretch (cent)				
$\langle 7, 11, 16|$	7.60	7.72	7.88	7.66	7.74	8.71	8.69	9.08	9.12	9.18
$\langle 12, 19, 28|$	2.94	2.96	3.04	3.02	3.04	-1.20	-1.28	-1.51	-1.35	-1.43
$\langle 15, 24, 35|$	4.83	4.79	4.67	4.75	4.71	-5.56	-5.40	-5.33	-5.65	-5.55
$\langle 19, 30, 44|$	1.99	1.98	1.95	1.96	1.95	2.45	2.40	2.39	2.51	2.48
$\langle 22, 35, 51|$	2.64	2.62	2.60	2.67	2.65	-1.04	-0.93	-0.76	-0.97	-0.88
$\langle 31, 49, 72|$	1.63	1.61	1.58	1.63	1.61	1.04	0.97	0.88	1.02	0.97
$\langle 34, 54, 79|$	1.07	1.05	1.02	1.05	1.04	-1.09	-1.05	-1.02	-1.10	-1.07
$\langle 41, 65, 95|$	1.19	1.20	1.24	1.22	1.23	0.58	0.61	0.70	0.64	0.68
$\langle 46, 73, 107|$	0.90	0.91	0.92	0.90	0.91	-1.11	-1.11	-1.14	-1.16	-1.16
$\langle 53, 84, 123|$	0.26	0.27	0.27	0.27	0.27	0.18	0.19	0.21	0.20	0.20

$$G(T_8 \cap P_5) = \begin{pmatrix} 1 \times 3 + 2^2 + 3^2 & 1 - 1 & 0 \\ 1 - 1 & 1 \times 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 9-limit only adds 9:1 so you can guess what happens there. The 10-limit is a bit more interesting because we have 5:2 as well as 10:9.

$$G(T_{10} \cap P_5) = \begin{pmatrix} 1 \times 5 + 2^2 + 3^2 & 1 - 1 & 1 - 1 \\ 1 - 1 & 1 \times 3 + 2^2 & 0 \\ 1 - 1 & 0 & 1 \times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The 12-limit gives us 12:1 and 4:3

$$G(T_{12} \cap P_5) = \begin{pmatrix} 26 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

To get a full 3:4:5 chord we have to go to the 20-limit

$$G(T_{20} \cap P_5) = \begin{pmatrix} 52 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Or the 40-limit to get 4:5:6

$$G(T_{40} \cap P_5) = \begin{pmatrix} 125 & 0 & 0 \\ 0 & 42 & 0 \\ 0 & 0 & 17 \end{pmatrix}$$

To get 5-limit harmony with any other inversions or four part harmony, you need at least the 80-limit

$$G(T_{80} \cap P_5) = \begin{pmatrix} 263 & 0 & 0 \\ 0 & 84 & 0 \\ 0 & 0 & 41 \end{pmatrix}$$

For extended 5-limit harmony including major seventh chords, you need the 120-limit

$$G(T_{120} \cap P_5) = \begin{pmatrix} 369 & 0 & 0 \\ 0 & 140 & 0 \\ 0 & 0 & 57 \end{pmatrix}$$

The 240-limit gives you some more inversions

$$G(T_{240} \cap P_5) = \begin{pmatrix} 694 & 0 & 0 \\ 0 & 250 & 0 \\ 0 & 0 & 110 \end{pmatrix}$$

The numbers are getting big again, so here are some

Table 4: Figures for some 5-prime limit rank 2 temperament classes (and the prime limit with Tenney weighting)

Product limit		20	40	80	120	240	Prime
Name	ETs	Scalar complexity					
Meantone	12&19	0.69	0.70	0.72	0.70	0.71	0.71
Srutal	12&34	1.17	1.17	1.18	1.19	1.20	1.22
Hanson	19&34	1.62	1.61	1.57	1.59	1.58	1.55
Helmholtz	12&53	1.69	1.71	1.75	1.74	1.75	1.79
Magic	19&22	1.42	1.40	1.37	1.41	1.40	1.40
Augmented	12&15	0.90	0.89	0.87	0.90	0.89	0.89
Tetracot	7&34	1.60	1.62	1.65	1.61	1.62	1.61
Sensipent	19&46	2.04	2.04	2.02	2.01	2.01	1.97
Amity	7&53	2.27	2.30	2.34	2.28	2.30	2.29
Wuerschmidt	31&34	2.32	2.29	2.24	2.31	2.28	2.29
Name	ETs	Scalar badness (centified)					
Meantone	12&19	1.11	1.11	1.12	1.12	1.12	1.12
Srutal	12&34	1.01	1.01	1.02	1.02	1.02	1.02
Hanson	19&34	0.42	0.42	0.42	0.42	0.42	0.42
Helmholtz	12&53	0.10	0.10	0.10	0.10	0.10	0.10
Magic	19&22	1.53	1.53	1.54	1.54	1.54	1.55
Augmented	12&15	2.12	2.12	2.13	2.14	2.14	2.15
Tetracot	7&34	1.43	1.43	1.44	1.44	1.44	1.45
Sensipent	19&46	0.69	0.69	0.70	0.70	0.70	0.70
Amity	7&53	0.32	0.32	0.32	0.32	0.32	0.32
Wuerschmidt	31&34	0.59	0.59	0.59	0.60	0.60	0.60
Name	ETs	Optimal error (cent/oct)					
Meantone	12&19	1.61	1.59	1.55	1.60	1.58	1.58
Srutal	12&34	0.86	0.87	0.86	0.85	0.85	0.84
Hanson	19&34	0.26	0.26	0.27	0.27	0.27	0.27
Helmholtz	12&53	0.06	0.06	0.06	0.06	0.06	0.06
Magic	19&22	1.07	1.09	1.12	1.09	1.10	1.11
Augmented	12&15	2.35	2.38	2.44	2.38	2.40	2.40
Tetracot	7&34	0.89	0.88	0.87	0.90	0.89	0.90
Sensipent	19&46	0.34	0.34	0.34	0.35	0.35	0.36
Amity	7&53	0.14	0.14	0.14	0.14	0.14	0.14
Wuerschmidt	31&34	0.25	0.26	0.27	0.26	0.26	0.26

normalized metrics for the above limits with full harmony.

$$G(T_{20} \cap P_5) = \begin{pmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.365 & 0.000 \\ 0.000 & 0.000 & 0.135 \end{pmatrix}$$

$$G(T_{40} \cap P_5) = \begin{pmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.336 & 0.000 \\ 0.000 & 0.000 & 0.136 \end{pmatrix}$$

$$G(T_{80} \cap P_5) = \begin{pmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.319 & 0.000 \\ 0.000 & 0.000 & 0.156 \end{pmatrix}$$

$$G(T_{120} \cap P_5) = \begin{pmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.379 & 0.000 \\ 0.000 & 0.000 & 0.154 \end{pmatrix}$$

$$G(T_{240} \cap P_5) = \begin{pmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.360 & 0.000 \\ 0.000 & 0.000 & 0.159 \end{pmatrix}$$

For comparison, the metric for Tenney weighting is

$$W^2 = \begin{pmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.398 & 0.000 \\ 0.000 & 0.000 & 0.185 \end{pmatrix}$$

The more realistic limits are close to this, but consistently give octaves more weight.

Table 3 shows the errors for some equal temperaments. This is the equivalent of the two Pythagorean tables. So you can compare the pure octave, optimal, and nor-

malized errors and see how far the octaves have to be stretched to reach the optimum.

Unlike the Pythagorean examples, the scale stretch is important here. For example, 22-equal is consistently better than 19-equal for pure octaves. But for optimized octaves 19-equal is consistently better than 22-equal. The reason is that the octaves are detuned more than twice as much in 19-equal.

Generally, the error normalization is doing its job. Although the raw errors get consistently larger the more intervals you look at, the normalized errors agree for a given equal temperament to about two figure accuracy.

The octave stretches look more variable. But the numbers in the table are the cents difference between a pure and stretched octave. The difference in the amount of optimal stretching is very small relative to the original size of the octave.

Table 4 gives the scalar complexity and badness and optimal error for each of your favorite 5-limit rank 2 temperament classes. If you don't recognize the temperament from the name you can get the mapping from the pair of equal temperaments, whose mappings are given in Table 3. Because I calculate the badness directly, I don't need the optimal tunings, so I don't tell you either.

The first section lists scalar complexities for the different temperament classes and different Tenney limits. The right hand column is the Tenney weighted prime complexity, or plain "scalar complexity" as defined in *Primerr*. The units are whatever you get by measuring intervals in octaves and applying the normalization.

The first section shows scalar badness, using the units you get by measuring intervals in octaves, normalizing, and then multiplying by twelve hundred. That's equivalent to complexity as above and errors in cents per octave. The right hand column is once again the way it was calculated in *Primerr*. Scalar badness on its own isn't very exciting, but it's one of the things you calculate to get the optimal error, so you may as well see how it behaves.

The last section gives the optimal errors. They're obtained by dividing the scalar badness by the scalar complexity. The right hand column is the TOP-RMS error. Units are in cents per octave.

After all that, Table 4 doesn't tell you anything very interesting. In all cases the different Tenney limit subsets agree with the Tenney-weighted prime measures to about two significant figures.

2.3 7-Prime Limits

There are four different Tenney limits that naturally fit the 7-prime limit. They're all useless, but still, here are

the metrics.

$$G(T_7) = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$G(T_8) = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$G(T_9) = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$G(T_{10}) = \begin{pmatrix} 18 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To play actual chords, it's nice to have 7:6 available, which means going up to the 42-limit

$$G(T_{42} \cap P_7) = \begin{pmatrix} 139 & 0 & 0 & 0 \\ 0 & 48 & 0 & 0 \\ 0 & 0 & 19 & 0 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$

and to round up to the octave you need an 8:7 from the 56-limit

$$G(T_{56} \cap P_7) = \begin{pmatrix} 193 & 0 & 0 & 0 \\ 0 & 76 & 0 & 0 \\ 0 & 0 & 29 & 0 \\ 0 & 0 & 0 & 19 \end{pmatrix}$$

Now, for the 9-limit, how about 9:8?

$$G(T_{72} \cap P_7) = \begin{pmatrix} 267 & 0 & 0 & 0 \\ 0 & 96 & 0 & 0 \\ 0 & 0 & 37 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}$$

Some more inversions, I hear you ask!

$$G(T_{144} \cap P_7) = \begin{pmatrix} 556 & 0 & 0 & 0 \\ 0 & 204 & 0 & 0 \\ 0 & 0 & 82 & 0 \\ 0 & 0 & 0 & 51 \end{pmatrix}$$

For some extended harmonies, maybe you want to go up to 21:20 for which you need

$$G(T_{420} \cap P_7) = \begin{pmatrix} 1462 & 0 & 0 & 0 \\ 0 & 561 & 0 & 0 \\ 0 & 0 & 246 & 0 \\ 0 & 0 & 0 & 158 \end{pmatrix}$$

Table 5: Errors and the like for some 7-prime limit equal temperaments

Product limit	42	56	72	144	420	42	56	72	144	420
Mapping	Pure octave RMS (cent)					Optimal RMS (cent)				
$\langle 10, 16, 23, 28|$	25.73	28.76	29.20	33.40	39.77	25.73	28.76	29.20	33.39	39.70
$\langle 12, 19, 28, 34|$	19.08	20.78	21.40	24.01	29.55	15.53	16.96	17.48	19.60	23.56
$\langle 15, 24, 35, 42|$	21.19	23.78	24.12	27.43	32.30	17.78	19.61	20.11	22.65	26.88
$\langle 19, 30, 44, 53|$	14.54	15.97	16.38	18.39	22.34	8.98	9.62	10.06	11.24	13.33
$\langle 22, 35, 51, 62|$	10.56	11.70	11.95	13.46	16.14	8.91	9.80	10.05	11.42	13.70
$\langle 27, 43, 63, 76|$	13.84	15.43	15.69	17.91	21.40	8.04	8.69	9.05	10.24	11.88
$\langle 31, 49, 72, 87|$	5.41	6.09	6.17	7.00	8.16	4.65	5.19	5.29	6.01	7.04
$\langle 41, 65, 95, 115|$	4.14	4.57	4.66	5.37	6.54	3.46	3.83	3.91	4.48	5.35
$\langle 46, 73, 107, 129|$	4.51	5.01	5.10	5.83	7.02	4.11	4.50	4.62	5.22	6.34
$\langle 53, 84, 123, 149|$	2.72	2.95	3.05	3.40	4.20	2.62	2.85	2.94	3.30	4.05
Mapping	Normalized optimum (cent/oct)					Octave stretch (cent)				
$\langle 10, 16, 23, 28|$	8.01	8.19	8.06	8.15	8.18	0.18	0.04	0.09	0.17	0.48
$\langle 12, 19, 28, 34|$	4.83	4.83	4.83	4.79	4.85	-3.44	-3.41	-3.40	-3.38	-3.67
$\langle 15, 24, 35, 42|$	5.53	5.59	5.55	5.53	5.54	-3.58	-3.82	-3.66	-3.77	-3.68
$\langle 19, 30, 44, 53|$	2.79	2.74	2.78	2.74	2.75	3.57	3.64	3.58	3.56	3.71
$\langle 22, 35, 51, 62|$	2.77	2.79	2.77	2.79	2.82	-1.77	-1.82	-1.78	-1.74	-1.76
$\langle 27, 43, 63, 76|$	2.50	2.48	2.50	2.50	2.45	-3.50	-3.62	-3.53	-3.58	-3.66
$\langle 31, 49, 72, 87|$	1.45	1.48	1.46	1.47	1.45	0.86	0.91	0.88	0.88	0.85
$\langle 41, 65, 95, 115|$	1.08	1.09	1.08	1.09	1.10	0.71	0.71	0.70	0.72	0.77
$\langle 46, 73, 107, 129|$	1.28	1.28	1.28	1.28	1.31	-0.58	-0.63	-0.60	-0.63	-0.62
$\langle 53, 84, 123, 149|$	0.81	0.81	0.81	0.80	0.83	-0.23	-0.22	-0.22	-0.21	-0.23

Some normalized metrics:

$$G(T_{42} \cap P_7) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.345 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.137 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.094 \end{pmatrix}$$

$$G(T_{56} \cap P_7) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.394 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.150 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.098 \end{pmatrix}$$

$$G(T_{72} \cap P_7) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.360 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.139 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.094 \end{pmatrix}$$

$$G(T_{144} \cap P_7) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.367 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.147 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.092 \end{pmatrix}$$

$$G(T_{420} \cap P_7) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.384 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.168 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.108 \end{pmatrix}$$

The metric for Tenney weighting:

$$G(W^2) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.398 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.185 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.127 \end{pmatrix}$$

Like with the 5-limit, the normalized metrics here all give more weight to octaves than the Tenney weighting of primes. I don't think that's an artifact of all the limits rounding off at an even number.

Tables 5 and 6 do for the 7-limit what Tables 3 and 4 did for the 5-limit They still aren't very interesting because the different Tenney limit subsets agree closely with the Tenney-weighted prime measures.

Of course, you may consider this agreement to be an interesting phenomenon given that the Tenney-limit metrics are consistently different to the Tenney-weighting. It's up to you.

Table 6: Figures for some 7-prime limit rank 2 temperament classes (and the prime limit with Tenney weighting)

	Product limit	42	56	72	144	420	Prime
Name	ETs	Scalar complexity					
Meantone	19&31	1.34	1.32	1.33	1.31	1.33	1.35
Magic	19&41	1.79	1.78	1.78	1.77	1.79	1.80
Orwell	22&31	2.24	2.27	2.25	2.26	2.26	2.26
Miracle	31&41	2.41	2.47	2.43	2.45	2.45	2.44
Garibaldi	12&41	2.31	2.31	2.31	2.30	2.32	2.34
Sensisept	19&27	1.99	1.95	1.98	1.97	1.93	1.89
Pajara	12&22	1.18	1.20	1.18	1.19	1.20	1.20
Myna	27&31	2.40	2.40	2.41	2.40	2.34	2.28
Augene	12&27	1.40	1.41	1.40	1.39	1.41	1.43
Catakleismic	19&53	2.85	2.80	2.83	2.80	2.83	2.87
Name	ETs	Scalar badness (centified)					
Meantone	19&31	1.89	1.89	1.89	1.87	1.87	1.87
Magic	19&41	1.90	1.89	1.90	1.90	1.92	1.93
Orwell	22&31	1.61	1.64	1.62	1.61	1.65	1.69
Miracle	31&41	1.30	1.31	1.30	1.32	1.29	1.26
Garibaldi	12&41	1.58	1.59	1.57	1.58	1.65	1.70
Sensisept	19&27	2.50	2.45	2.48	2.46	2.48	2.50
Pajara	12&22	2.95	2.98	2.95	2.94	3.02	3.08
Myna	27&31	2.23	2.25	2.24	2.27	2.23	2.17
Augene	12&27	3.26	3.23	3.25	3.22	3.21	3.19
Catakleismic	19&53	1.38	1.35	1.37	1.35	1.37	1.38
Name	ETs	Optimal error (cent/oct)					
Meantone	19&31	1.41	1.43	1.42	1.43	1.40	1.38
Magic	19&41	1.06	1.07	1.06	1.07	1.08	1.07
Orwell	22&31	0.72	0.72	0.72	0.71	0.73	0.75
Miracle	31&41	0.54	0.53	0.54	0.54	0.53	0.51
Garibaldi	12&41	0.68	0.69	0.68	0.69	0.71	0.73
Sensisept	19&27	1.25	1.26	1.25	1.25	1.29	1.32
Pajara	12&22	2.50	2.48	2.49	2.47	2.52	2.57
Myna	27&31	0.93	0.94	0.93	0.94	0.95	0.95
Augene	12&27	2.33	2.29	2.32	2.32	2.28	2.23
Catakleismic	19&53	0.48	0.48	0.48	0.48	0.48	0.48

2.4 11-Prime Limits

Let's go straight to some chordal 11-limit metrics, shall we? To get 11:10 you need the 110-limit

Then the limits get successively higher for 12:11, 14:11, 15:11, and 16:11.

$$G(T_{132} \cap P_{11}) = \begin{pmatrix} 560 & 0 & 0 & 0 & 0 \\ 0 & 196 & 0 & 0 & 0 \\ 0 & 0 & 82 & 0 & 0 \\ 0 & 0 & 0 & 49 & 0 \\ 0 & 0 & 0 & 0 & 31 \end{pmatrix}$$

$$G(T_{110} \cap P_{11}) = \begin{pmatrix} 423 & 0 & 0 & 0 & 0 \\ 0 & 172 & 0 & 0 & 0 \\ 0 & 0 & 69 & 0 & 0 \\ 0 & 0 & 0 & 43 & 0 \\ 0 & 0 & 0 & 0 & 23 \end{pmatrix}$$

$$G(T_{154} \cap P_{11}) = \begin{pmatrix} 616 & 0 & 0 & 0 & 0 \\ 0 & 228 & 0 & 0 & 0 \\ 0 & 0 & 104 & 0 & 0 \\ 0 & 0 & 0 & 65 & 0 \\ 0 & 0 & 0 & 0 & 35 \end{pmatrix}$$

Table 7: Errors and the like for some 11-prime limit equal temperaments

Product limit	110	176	220	352	110	176	220	352
Mapping	Pure octave RMS (cent)				Optimal RMS (cent)			
$\langle 12, 19, 28, 34, 42|$	30.65	33.57	34.12	37.42	23.10	25.03	25.63	27.70
$\langle 14, 22, 32, 39, 48|$	44.93	48.65	49.58	53.67	24.73	27.11	27.49	29.70
$\langle 15, 24, 35, 42, 52|$	25.15	26.50	27.51	29.16	20.48	21.72	22.51	24.09
$\langle 19, 30, 44, 53, 65|$	26.36	28.87	29.36	32.06	16.94	18.73	18.95	20.64
$\langle 19, 30, 44, 53, 66|$	18.31	19.68	20.23	22.00	15.73	17.20	17.52	19.15
$\langle 22, 35, 51, 62, 76|$	12.47	13.23	13.71	14.74	11.41	12.27	12.59	13.60
$\langle 27, 43, 63, 76, 94|$	20.02	21.68	22.14	23.91	10.59	11.63	11.82	12.77
$\langle 31, 49, 72, 87, 107|$	7.59	8.08	8.35	8.89	5.99	6.43	6.60	7.07
$\langle 41, 65, 95, 115, 142|$	5.30	5.79	5.86	6.39	5.03	5.50	5.57	6.06
$\langle 46, 73, 107, 129, 159|$	5.50	5.93	6.06	6.54	5.27	5.70	5.82	6.32
Mapping	Normalized optimum (cent/oct)				Octave stretch (cent)			
$\langle 12, 19, 28, 34, 42|$	5.82	5.83	5.84	5.84	-5.06	-5.19	-5.11	-5.28
$\langle 14, 22, 32, 39, 48|$	6.23	6.32	6.26	6.26	9.53	9.48	9.47	9.50
$\langle 15, 24, 35, 42, 52|$	5.16	5.06	5.13	5.08	-3.67	-3.53	-3.59	-3.45
$\langle 19, 30, 44, 53, 65|$	4.27	4.36	4.32	4.35	5.11	5.14	5.13	5.20
$\langle 19, 30, 44, 53, 66|$	3.96	4.01	3.99	4.04	2.37	2.23	2.31	2.29
$\langle 22, 35, 51, 62, 76|$	2.88	2.86	2.87	2.87	-1.27	-1.15	-1.23	-1.20
$\langle 27, 43, 63, 76, 94|$	2.67	2.71	2.69	2.69	-4.27	-4.25	-4.25	-4.25
$\langle 31, 49, 72, 87, 107|$	1.51	1.50	1.50	1.49	1.17	1.14	1.16	1.14
$\langle 41, 65, 95, 115, 142|$	1.27	1.28	1.27	1.28	0.42	0.42	0.41	0.43
$\langle 46, 73, 107, 129, 159|$	1.33	1.33	1.33	1.33	-0.40	-0.38	-0.38	-0.36

$$G(T_{165} \cap P_{11}) = \begin{pmatrix} 668 & 0 & 0 & 0 & 0 \\ 0 & 264 & 0 & 0 & 0 \\ 0 & 0 & 110 & 0 & 0 \\ 0 & 0 & 0 & 65 & 0 \\ 0 & 0 & 0 & 0 & 39 \end{pmatrix}$$

That's probably what you'd use to represent classic 11-limit harmony. But maybe you want to push it a bit higher to 20:11.

$$G(T_{220} \cap P_{11}) = \begin{pmatrix} 896 & 0 & 0 & 0 & 0 \\ 0 & 346 & 0 & 0 & 0 \\ 0 & 0 & 142 & 0 & 0 \\ 0 & 0 & 0 & 89 & 0 \\ 0 & 0 & 0 & 0 & 49 \end{pmatrix}$$

Or even 32:11?

$$G(T_{352} \cap P_{11}) = \begin{pmatrix} 1462 & 0 & 0 & 0 & 0 \\ 0 & 537 & 0 & 0 & 0 \\ 0 & 0 & 236 & 0 & 0 \\ 0 & 0 & 0 & 152 & 0 \\ 0 & 0 & 0 & 0 & 83 \end{pmatrix}$$

Now for the normalized equivalents of those metrics.

$$G(T_{110} \cap P_{11}) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.407 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.163 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.102 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.054 \end{pmatrix}$$

$$G(T_{132} \cap P_{11}) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.350 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.146 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.087 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.055 \end{pmatrix}$$

$$G(T_{154} \cap P_{11}) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.370 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.169 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.106 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.057 \end{pmatrix}$$

$$G(T_{165} \cap P_{11}) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.395 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.165 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.097 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.058 \end{pmatrix}$$

Table 8: Figures for some 11-prime limit rank 2 temperament classes (and the prime limit with Tenney weighting)

Product limit		110	176	220	352	Prime
Name	ETs	Scalar complexity				
Orwell	22&31	2.13	2.10	2.11	2.09	2.05
Meantone	12&31	1.83	1.85	1.84	1.85	1.92
Miracle	31&41	2.73	2.73	2.73	2.72	2.79
Valentine	15&31	2.41	2.36	2.39	2.36	2.31
Myna	27&31	2.62	2.64	2.64	2.63	2.61
Magic	22&41	2.22	2.23	2.23	2.24	2.35
Squares	14&31	2.27	2.30	2.28	2.28	2.20
Meanpop	19a&31	2.19	2.22	2.20	2.24	2.42
Pajara	12&22	1.57	1.57	1.57	1.56	1.61
Augene	12&15	1.45	1.43	1.44	1.43	1.45
Name	ETs	Scalar badness (centified)				
Orwell	22&31	2.26	2.25	2.26	2.26	2.36
Meantone	12&31	2.72	2.71	2.72	2.71	2.76
Miracle	31&41	1.39	1.39	1.39	1.38	1.35
Valentine	15&31	2.38	2.36	2.37	2.36	2.39
Myna	27&31	2.30	2.31	2.30	2.29	2.22
Magic	22&41	2.71	2.73	2.71	2.73	2.88
Squares	14&31	3.37	3.39	3.37	3.34	3.19
Meanpop	19a&31	2.89	2.88	2.89	2.89	2.99
Pajara	12&22	3.59	3.58	3.59	3.61	3.71
Augene	12&15	3.86	3.86	3.88	3.86	3.84
Name	ETs	Optimal error (cent/oct)				
Orwell	22&31	1.06	1.07	1.07	1.08	1.15
Meantone	12&31	1.49	1.46	1.48	1.46	1.44
Miracle	31&41	0.51	0.51	0.51	0.51	0.48
Valentine	15&31	0.99	1.00	0.99	1.00	1.03
Myna	27&31	0.88	0.88	0.87	0.87	0.85
Magic	22&41	1.22	1.22	1.22	1.22	1.23
Squares	14&31	1.48	1.47	1.48	1.46	1.45
Meanpop	19a&31	1.32	1.30	1.31	1.29	1.24
Pajara	12&22	2.29	2.29	2.29	2.31	2.30
Augene	12&15	2.66	2.71	2.69	2.69	2.65

$$G(T_{220}\cap P_{11}) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.386 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.158 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.099 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.055 \end{pmatrix}$$

Compare them with the Tenney weighting of primes.

$$G(W^2) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.398 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.185 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.127 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.084 \end{pmatrix}$$

The other metrics still tend to give more weight to the octaves, although the 110-limit does give $3{:}1$ a little more weight than the Tenney weighting. Also, the other metrics deviate from each other less than they all deviate from the Tenney weighting.

Tables 7 and 8 show the errors and so on for some 11-limit rank 1 and 2 temperament classes. They're much like

$$G(T_{352}\cap P_{11}) = \begin{pmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.367 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.161 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.104 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.057 \end{pmatrix}$$

the 5- and 7-limit tables in the boring agreement of the different Tenney limits with each other and the Tenney-weighted prime limit. The only interesting detail is that Table 7 contains two mappings for 19-equal. So that you know which one is used for which rank 2 temperament I've arbitrarily called the second on in the list $19a$ in Table 8. That is, 19a refers to $\langle 19, 30, 44, 53, 66|$.

2.5 General Rules

It surely can't have escaped your notice that all the metrics for equally weighted Tenney limits have their only non-zero elements on the diagonal. That means Theorem 2 can be generalized for any prime limit.

Theorem 3 *For any Tenney limit, the off-diagonal elements of the metric are always zero. Hence the unweighted RMS of the subset of the Tenney limit is the same as a weighted RMS of the prime limit.*

The proof follows from Lemma 1 and a generalization of Lemma 2.

Lemma 3 *Consider an interval with at least two prime factors, p and q, with exponents a and b respectively. The interval also involves a rational factor x with no factors of p or q. Such an interval comes in two forms within a given Tenney limit: $xp^a{:}q^b$ and $xp^aq^b{:}1$ where p and q are chosen so that $xp^a > q^b$. Factors of $1{:}x$ can be considered separately.*

This is true for exactly the same reasons Lemma 2 was true.

It doesn't matter how many times the other primes are involved, the two off-diagonals that involve p and q will always be zero because intervals that use them both come in pairs that cancel out. And the same is true for all other pairs of primes.

I also said that the normalized Pythagorean metrics tend towards Tenney weighting. I believe the same is true for any limit.

Conjecture 1 *The normalized metric for an unweighted Tenney limit tends towards the metric for a Tenney weighted prime limit as the Tenney limit approaches infinity.*

I still can't prove it.

It'd be nice to come up with a proof, but it isn't that important. You can verify it sort of works up to any product limit you like. If something weird happens at limits that are musically useless it doesn't affect the validity of an observation at lower limits. So it's more important that, at least in the 11-limit, the metrics are consistently different from Tenney weighting.

Rule of Thumb 1 *A relatively small Tenney limit will tend to give more weight to lower primes than the Tenney weighting of primes. What counts as "relatively small" grows with the number of primes you consider.*

Despite this, Tenney weighting to approximate a Tenney limit is still a pretty good bet, as you'd expect from the names.

3 Cross-Weighted Metrics

3.1 Cross-Weighting

The following general class of metrics gives you a bit more flexibility than simply weighting the prime intervals without the mess of defining all those composite intervals.

$$G(\lambda) = W^2 - \lambda \frac{W^2 H H^T W^2}{H^T W^2 H} \qquad (16)$$

Here, λ is a cross-weighting term that spreads the weighting areound different primes. It's somewhere in the region $0 \leq \lambda \leq 1$. $G(0) = W^2$ is the metric for weighted prime errors. When λ gets larger, smaller intervals become more important.

The reasons for using this metric will hopefully become apparent later on. For now, let's just get the theory out of the way. When the examples demand this modification I'm going to need the language to talk about it.

To get complexity and badness in terms of this class of metric, let's first look at how the normalization constant comes out.

$$\begin{aligned}
H^T G(\lambda) H &= H^T \left[W^2 - \lambda \frac{W^2 H H^T W^2}{H^T W^2 H} \right] H \\
&= H^T W^2 H - \lambda \frac{H^T W^2 H H^T W^2 H}{H^T W^2 H} \\
&= H^T W^2 H - \lambda (H^T W^2 H) \\
&= (1 - \lambda) H^T W^2 H \qquad (17)
\end{aligned}$$

The formula for complexity is then straightforward to find and not particularly interesting.

$$\begin{aligned}
k^2 &= \left| \frac{M^T G(\lambda) M}{H^T G(\lambda) H} \right| \\
&= \left| \frac{M^T \left(W^2 - \lambda \frac{W^2 H H^T W^2}{H^T W^2 H} \right) M}{(1 - \lambda) H^T W^2 H} \right| \\
&= \left| \frac{M^T W^2 M - \lambda \frac{M^T W^2 H H^T W^2 M}{H^T W^2 H}}{(1 - \lambda) H^T W^2 H} \right| \qquad (18)
\end{aligned}$$

There's one more term we need for the badness.

$$G(\lambda)^T H = \left(W^2 - \lambda \frac{W^2 H H^T W^2}{H^T W^2 H} \right) H$$

$$= W^2 H - \lambda \frac{W^2 H H^T W^2 H}{H^T W^2 H}$$

$$= W^2 H - \lambda (W^2 H)$$

$$= (1 - \lambda) W^2 H$$

$$G(\lambda)^T H H^T G(\lambda) = (1 - \lambda) W^2 H \left[(1 - \lambda) W^2 H \right]^T$$

$$= (1 - \lambda)^2 W^2 H H^T W^2 \tag{19}$$

Combine that with Equation 17 to get

$$\frac{G(\lambda)^T H H^T G(\lambda)}{H^T G(\lambda) H} = \frac{(1 - \lambda)^2 W^2 H H^T W^2}{(1 - \lambda) H^T W^2 H}$$

$$= (1 - \lambda) \frac{W^2 H H^T W^2}{H^T W^2 H} \tag{20}$$

Equation 14, giving the badness we're looking for, can be written as

$$B^2 = \left| \frac{M^T \left[G(\lambda) - \frac{G(\lambda) H H^T G(\lambda)}{H^T G(\lambda) H} \right] M}{H^T G(\lambda) H} \right| \tag{21}$$

Substituting in Equations 16 and 20 gives

$$G(\lambda) - \frac{G(\lambda) H H^T G(\lambda)}{H^T G(\lambda) H}$$

$$= W^2 - \lambda \frac{W^2 H H^T W^2}{H^T W^2 H} - (1 - \lambda) \frac{W^2 H H^T W^2}{H^T W^2 H}$$

$$= W^2 - (\lambda + 1 - \lambda) \frac{W^2 H H^T W^2}{H^T W^2 H}$$

$$= W^2 - \frac{W^2 H H^T W^2}{H^T W^2 H} \tag{22}$$

At last, we have a badness formula!

$$B^2 = \left| \frac{M^T W^2 M - \frac{M^T W^2 H H^T W^2 M}{H^T W^2 H}}{(1 - \lambda) H^T W^2 H} \right| \tag{23}$$

3.2 Renormalized Badness and Complexity

Equation 23 has a strange property in that, if you write badness as a function of λ, it's a very simple function.

$$B(\lambda) = \frac{B(0)}{(1 - \lambda)^{\frac{r}{2}}} \tag{24}$$

Because the $1 - \lambda$ term was inside the determinant, it gets raised to the power of r, the rank of the temperament class, as well as being square rooted.

The point of λ is that it's a small perturbation; it shouldn't affect the badness this directly. So, let's renormalize badness so that $B_{\text{ren}} = B(0)$ always.

$$B_{\text{ren}}^2 = \left| (1 - \lambda) \left(\frac{M^T G M}{H^T G H} - \frac{M^T G H H^T G M}{H^T G H H^T G H} \right) \right| \tag{25}$$

$$= \left| \frac{M^T W^2 M}{H^T W^2 H} - \frac{M^T W^2 H H^T W^2 M}{H^T W^2 H H^T W^2 H} \right| \tag{26}$$

The complexity also has to be renormalized to bring it into line.

$$k_{\text{ren}}^2 = \left| (1 - \lambda) \frac{M^T G(\lambda) M}{H^T G(\lambda) H} \right| \tag{27}$$

$$= \left| \frac{M^T W^2 M}{H^T W^2 H} - \lambda \frac{M^T W^2 H H^T W^2 M}{H^T W^2 H H^T W^2 H} \right| \tag{28}$$

Note that Equation 26 is a special case of Equation 28 with $\lambda = 1$. Also, if you take at look at *Primerr*, you may see that Equation 28 here is essentially the same as Equation 82 on p. 18 of *Primerr*.

You can re-write Equation 27 using the identity

$$1 - \lambda = \frac{H^T G(\lambda) H}{H^T W^2 H} \tag{29}$$

to give

$$k_{\text{ren}}^2 = \left| \frac{M^T G(\lambda) M}{H^T W^2 H} \right| \tag{30}$$

3.3 Renormalized Error and Complexity

The renormalized badness in the previous section suggests that $H^T W^2 H$ is a better normalization factor than $H^T G H$. However, the normalizations of badness and complexity cancel out when you calculate the error. The error is also normalized by $H^T G H$ in Equation 12. To renormalize it in terms of $H^T W^2 H$ and preserve the badness as error times complexity, we need to adjust either complexity or badness by a factor of $1 - \lambda$. Because badness is already independent of λ, let's re-define complexity.

$$k_{\text{ren}}^2 = \frac{1}{1 - \lambda} \left| \frac{M^T G(\lambda) M}{H^T W^2 H} \right| \tag{31}$$

which is the same as

$$k_{\text{ren}}^2 = \frac{H^T W^2 H}{H^T G(\lambda) H} \left| \frac{M^T G(\lambda) M}{H^T W^2 H} \right| \tag{32}$$

The factors of $H^T W^2 H$ don't cancel out because one of them's inside the determinant.

3.4 General Renormalization

Complexity and badness as in Equation 32 and Equation 26 can be generalized for any metric G instead of $G(\lambda)$. The badness formula is

$$B^2(G, W) = \left| \frac{M^T G M}{H^T W^2 H} - \lambda \frac{M^T G H H^T G M}{H^T G H H^T W^2 H} \right| \quad (33)$$

However, this means finding a suitable W^2.

So far, the best I can think of is to make the diagonals of W^2 the eigenvalues of G. As long as the prime intervals are in ascending order, you can guess that the diagonals of W^2 should be as well. There must be a better way but this method is good enough for now.

It may appear that all this business of renormalization is about tweaking the formulas to get a specific result. After all, a formula can do whatever you like if it has enough free parameters. There is a bit of that here but notice that the normalizations only multiply each complexity, error, or badness by a factor that depends on H and G. If you're comparing the complexity, error, or badness of temperament classes with the same H and G no amount of renormalization can bias the result.

4 Equally Weighted Farey Limits

The Farey[7] f-limit contains all intervals with ratios $n{:}d$ in their lowest terms where $1 \le d \le n \le f$. In musical terms you can think of this as being all the intervals within the first f partials of a harmonic series.

I'll write the Farey f-limit as F_f. Like with Tenney limits, Farey limits can intersect with prime limits.

4.1 Pythagorean Limits

The simplest limit with 2 and 3 is, of course, the 3-limit. As a Farey limit it contains the intervals 1:1, 2:1, 3:2, and 3:1. Notice that, unlike the Tenney limits for such small numbers, we already have an interval smaller than an octave. The 3-limit metric is

$$G(F_3) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Again unlike the Tenney limit, this has non-zero off-diagonals. The 3:2 contributes a factor of -1×1 and has no 6:1 to balance it.

Even the strictest Pythagorean will expect an interval of $4{:}3$. The 4-limit gives you that, along with $4{:}1$. The metric is

$$G(F_4) = \begin{pmatrix} 10 & -3 \\ -3 & 3 \end{pmatrix}$$

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[7]As in a Farey sequence, which is a standard mathematical term. See, for example, Smith 2001.

These intervals are already sufficient for Pythagorean harmony. But let's extend it a bit anyway.

$$G(F_6 \cap P_3) = \begin{pmatrix} 11 & -2 \\ -2 & 4 \end{pmatrix}$$

$$G(F_8 \cap P_3) = \begin{pmatrix} 29 & -5 \\ -5 & 5 \end{pmatrix}$$

$$G(F_9 \cap P_3) = \begin{pmatrix} 43 & -17 \\ -17 & 21 \end{pmatrix}$$

$$G(F_{12} \cap P_3) = \begin{pmatrix} 47 & -15 \\ -15 & 22 \end{pmatrix}$$

$$G(F_{16} \cap P_3) = \begin{pmatrix} 95 & -27 \\ -27 & 27 \end{pmatrix}$$

$$G(F_{18} \cap P_3) = \begin{pmatrix} 96 & -25 \\ -25 & 31 \end{pmatrix}$$

$$G(F_{24} \cap P_3) = \begin{pmatrix} 105 & -22 \\ -22 & 32 \end{pmatrix}$$

$$G(F_{27} \cap P_3) = \begin{pmatrix} 135 & -52 \\ -52 & 77 \end{pmatrix}$$

$$G(F_{32} \cap P_3) = \begin{pmatrix} 235 & -82 \\ -82 & 91 \end{pmatrix}$$

The off-diagonal elements clearly aren't going away. Notice that they're negative, which means these metrics are of the general form I talked about in the last section. You'd almost think I planned it that way.

Here are the normalized metrics:

$$G(F_3) = \begin{pmatrix} 1.000 & -0.500 \\ -0.500 & 1.000 \end{pmatrix}$$

$$G(F_4) = \begin{pmatrix} 1.000 & -0.300 \\ -0.300 & 0.300 \end{pmatrix}$$

$$G(F_6 \cap P_3) = \begin{pmatrix} 1.000 & -0.182 \\ -0.182 & 0.364 \end{pmatrix}$$

$$G(F_8 \cap P_3) = \begin{pmatrix} 1.000 & -0.172 \\ -0.172 & 0.172 \end{pmatrix}$$

$$G(F_9 \cap P_3) = \begin{pmatrix} 1.000 & -0.395 \\ -0.395 & 0.488 \end{pmatrix}$$

$$G(F_{12} \cap P_3) = \begin{pmatrix} 1.000 & -0.319 \\ -0.319 & 0.468 \end{pmatrix}$$

$$G(F_{16} \cap P_3) = \begin{pmatrix} 1.000 & -0.284 \\ -0.284 & 0.284 \end{pmatrix}$$

$$G(F_{18} \cap P_3) = \begin{pmatrix} 1.000 & -0.260 \\ -0.260 & 0.323 \end{pmatrix}$$

Table 9: Errors and the like for some 5-prime limit equal temperaments

Farey limit	3	4	5	6	3	4	5	6
Mapping	Pure octave RMS (cent)				Optimal RMS (cent)			
$\langle 7, 11, 16 |$	13.26	12.58	28.28	27.31	8.34	11.78	21.62	22.17
$\langle 12, 19, 28 |$	1.60	1.51	9.53	9.85	1.00	1.41	7.95	9.23
$\langle 15, 24, 35 |$	14.73	13.98	13.16	13.15	9.11	13.01	11.13	10.16
$\langle 19, 30, 44 |$	5.89	5.59	5.95	5.81	3.69	5.23	4.81	4.36
$\langle 22, 35, 51 |$	5.83	5.53	6.22	6.97	3.62	5.16	6.12	6.95
$\langle 31, 49, 72 |$	4.23	4.01	3.62	4.05	2.65	3.75	3.62	3.91
$\langle 34, 54, 79 |$	3.21	3.04	2.61	2.71	2.00	2.84	2.37	2.25
$\langle 41, 65, 95 |$	0.40	0.37	3.98	4.07	0.25	0.35	3.28	3.78
$\langle 46, 73, 107 |$	1.95	1.85	3.31	3.18	1.22	1.73	2.50	2.47
$\langle 53, 84, 123 |$	0.06	0.05	0.93	0.93	0.03	0.05	0.74	0.84
Mapping	Normalized optimum (cent/oct)				Octave stretch (cent)			
$\langle 7, 11, 16 |$	6.15	6.62	10.55	10.42	9.17	3.49	13.91	11.24
$\langle 12, 19, 28 |$	0.74	0.79	3.88	4.34	1.10	0.43	-3.96	-2.39
$\langle 15, 24, 35 |$	6.72	7.31	5.43	4.78	-10.13	-4.02	-5.27	-5.80
$\langle 19, 30, 44 |$	2.72	2.94	2.35	2.05	4.07	1.57	2.65	2.68
$\langle 22, 35, 51 |$	2.67	2.89	2.99	3.27	-4.01	-1.57	0.86	-0.36
$\langle 31, 49, 72 |$	1.95	2.11	1.76	1.84	2.92	1.13	0.09	0.73
$\langle 34, 54, 79 |$	1.47	1.59	1.16	1.06	-2.21	-0.86	-0.83	-1.05
$\langle 41, 65, 95 |$	0.18	0.20	1.60	1.78	-0.27	-0.11	1.70	1.06
$\langle 46, 73, 107 |$	0.90	0.97	1.22	1.16	-1.35	-0.52	-1.64	-1.39
$\langle 53, 84, 123 |$	0.03	0.03	0.36	0.39	0.04	0.01	0.42	0.29

$$G(F_{24} \cap P_3) = \begin{pmatrix} 1.000 & -0.210 \\ -0.210 & 0.305 \end{pmatrix}$$

$$G(F_{27} \cap P_3) = \begin{pmatrix} 1.000 & -0.385 \\ -0.385 & 0.570 \end{pmatrix}$$

$$G(F_{32} \cap P_3) = \begin{pmatrix} 1.000 & -0.349 \\ -0.349 & 0.387 \end{pmatrix}$$

This sequence looks like it's tending towards some kind of stability. Maybe with the diagonals as Tenney weighting and the off-diagonals as the negation of the 3 weighting.

I didn't produce any tables for Pythagorean Farey limits, but I sneaked some Pythagorean columns into Table 9. You can see that the order of the errors is still preserved for the two different limits. However the normalization isn't doing such a good job of making the errors come out as nearly the same numbers.

There are also significant differences in the optimal octave stretch for the different Farey limits. That means if you care about optimizing for such a limit you do need to specify exactly which limit you care about.

4.2 5-Prime Limits

The metric for the Farey 5-limit is

$$G(F_5) = \begin{pmatrix} 15 & -3 & -3 \\ -3 & 4 & -1 \\ -3 & -1 & 4 \end{pmatrix}$$

It gives very basic 5-limit harmony involving 3:4:5 triads. To get a 4:5:6 we obviously need a 6.

$$G(F_6) = \begin{pmatrix} 17 & -1 & -4 \\ -1 & 6 & -2 \\ -4 & -2 & 5 \end{pmatrix}$$

If you want other inversions it's nice to have an 8 as well.

$$G(F_8 \cap P_5) = \begin{pmatrix} 44 & -4 & -7 \\ -4 & 7 & -2 \\ -7 & -2 & 6 \end{pmatrix}$$

That's it for 5-odd limit harmony. You may like to use even numbers higher than 8, but there's no 5-prime Farey limit that includes them without letting in higher odd numbers as well. But at least Farey limits do better than Tenney limits in containing an odd limit.

Table 10: Figures for some 5-prime limit rank 2 temperament classes (and the prime limit with Tenney weighting)

	Farey limit	5	6	8	10	12	15	16	Prime
Name	ETs			Scalar complexity					
Meantone	12&19	0.96	0.99	0.97	0.77	0.78	0.83	0.85	0.71
Srutal	12&34	1.44	1.63	1.36	1.54	1.51	1.40	1.23	1.22
Hanson	19&34	1.84	1.61	1.74	2.04	1.92	1.93	2.08	1.55
Helmholtz	12&53	2.25	2.51	2.20	2.10	2.10	2.04	1.84	1.79
Magic	19&22	1.52	1.48	1.36	1.92	1.81	1.70	1.68	1.40
Augmented	12&15	0.97	0.99	0.86	1.23	1.16	1.08	1.03	0.89
Tetracot	7&34	2.18	2.11	2.20	1.78	1.77	1.93	2.09	1.61
Sensipent	19&46	2.52	2.23	2.46	2.41	2.31	2.44	2.70	1.97
Amity	7&53	3.12	3.08	3.15	2.50	2.51	2.73	2.92	2.29
Wuerschmidt	31&34	2.49	2.46	2.22	3.14	2.97	2.77	2.70	2.29
Name	ETs			Scalar badness (centified)					
Meantone	12&19	1.66	1.64	1.50	1.67	1.60	1.60	1.55	1.12
Srutal	12&34	1.51	1.49	1.36	1.52	1.45	1.45	1.41	1.02
Hanson	19&34	0.63	0.62	0.56	0.63	0.60	0.60	0.58	0.42
Helmholtz	12&53	0.15	0.15	0.14	0.15	0.15	0.15	0.14	0.10
Magic	19&22	2.29	2.26	2.06	2.30	2.20	2.20	2.13	1.55
Augmented	12&15	3.17	3.13	2.85	3.20	3.05	3.05	2.95	2.15
Tetracot	7&34	2.14	2.11	1.92	2.15	2.05	2.05	1.99	1.45
Sensipent	19&46	1.04	1.02	0.93	1.04	0.99	1.00	0.96	0.70
Amity	7&53	0.48	0.47	0.43	0.48	0.46	0.46	0.44	0.32
Wuerschmidt	31&34	0.88	0.87	0.80	0.89	0.85	0.85	0.82	0.60
Name	ETs			Optimal error (cent/oct)					
Meantone	12&19	1.72	1.65	1.54	2.18	2.05	1.92	1.82	1.58
Srutal	12&34	1.05	0.92	1.00	0.99	0.96	1.03	1.15	0.84
Hanson	19&34	0.34	0.38	0.32	0.31	0.31	0.31	0.28	0.27
Helmholtz	12&53	0.07	0.06	0.06	0.07	0.07	0.07	0.08	0.06
Magic	19&22	1.50	1.52	1.51	1.20	1.21	1.30	1.27	1.11
Augmented	12&15	3.27	3.18	3.30	2.60	2.62	2.83	2.86	2.40
Tetracot	7&34	0.98	1.00	0.88	1.21	1.16	1.06	0.95	0.90
Sensipent	19&46	0.41	0.46	0.38	0.43	0.43	0.41	0.36	0.36
Amity	7&53	0.15	0.15	0.14	0.19	0.18	0.17	0.15	0.14
Wuerschmidt	31&34	0.36	0.35	0.36	0.28	0.29	0.31	0.30	0.26

Here are some more metrics for extended 5-limit harmony

$$G(F_9 \cap P_5) = \begin{pmatrix} 58 & -16 & -7 \\ -16 & 27 & -4 \\ -7 & -4 & 7 \end{pmatrix}$$

$$G(F_{10} \cap P_5) = \begin{pmatrix} 61 & -19 & -4 \\ -19 & 32 & -7 \\ -4 & -7 & 10 \end{pmatrix}$$

$$G(F_{12} \cap P_5) = \begin{pmatrix} 69 & -15 & -6 \\ -15 & 34 & -8 \\ -6 & -8 & 11 \end{pmatrix}$$

$$G(F_{16} \cap P_5) = \begin{pmatrix} 163 & -37 & -20 \\ -37 & 44 & -3 \\ -20 & -3 & 17 \end{pmatrix}$$

Here are the normalized metrics

$$G(F_5) = \begin{pmatrix} 1.000 & -0.200 & -0.200 \\ -0.200 & 0.267 & -0.067 \\ -0.200 & -0.067 & 0.267 \end{pmatrix}$$

$$G(F_6) = \begin{pmatrix} 1.000 & -0.059 & -0.235 \\ -0.059 & 0.353 & -0.118 \\ -0.235 & -0.118 & 0.294 \end{pmatrix}$$

$$G(F_8 \cap P_5) = \begin{pmatrix} 1.000 & -0.091 & -0.159 \\ -0.091 & 0.159 & -0.045 \\ -0.159 & -0.045 & 0.136 \end{pmatrix}$$

Table 11: Errors and the like for some 7-prime limit equal temperaments

Farey limit	7	8	9	10	7	8	9	10
Mapping	Pure octave RMS (cent)				Optimal RMS (cent)			
$\langle 10, 16, 23, 28|$	23.39	22.25	28.23	30.26	22.90	21.91	27.83	30.25
$\langle 12, 19, 28, 34|$	19.53	19.08	18.51	18.22	15.87	17.05	17.75	17.28
$\langle 15, 24, 35, 42|$	15.51	14.92	21.44	20.96	15.44	14.92	20.32	19.86
$\langle 19, 30, 44, 53|$	11.77	11.80	11.95	11.58	8.44	10.28	9.85	9.88
$\langle 22, 35, 51, 62|$	9.13	8.88	10.23	10.83	8.26	8.45	9.05	10.38
$\langle 27, 43, 63, 76|$	8.41	8.60	10.55	10.25	6.13	7.73	8.69	8.19
$\langle 31, 49, 72, 87|$	3.60	3.45	5.74	5.83	3.54	3.43	5.30	5.62
$\langle 41, 65, 95, 115|$	3.76	3.68	3.60	3.92	3.33	3.46	3.54	3.74
$\langle 46, 73, 107, 129|$	4.17	4.02	4.31	4.43	4.17	4.02	4.26	4.29
$\langle 53, 84, 123, 149|$	3.09	2.98	2.80	2.87	2.85	2.83	2.72	2.85
Mapping	Normalized optimum (cent/oct)				Octave stretch (cent)			
$\langle 10, 16, 23, 28|$	9.34	8.19	9.52	10.21	3.27	2.58	-3.03	-0.25
$\langle 12, 19, 28, 34|$	6.47	6.37	6.07	5.83	-7.76	-5.67	-3.38	-3.61
$\langle 15, 24, 35, 42|$	6.30	5.58	6.95	6.70	-1.00	-0.08	-4.40	-4.19
$\langle 19, 30, 44, 53|$	3.44	3.84	3.37	3.34	5.65	3.86	4.39	3.79
$\langle 22, 35, 51, 62|$	3.37	3.16	3.09	3.50	-2.65	-1.82	-3.07	-1.93
$\langle 27, 43, 63, 76|$	2.50	2.89	2.97	2.76	-3.93	-2.50	-3.85	-3.84
$\langle 31, 49, 72, 87|$	1.44	1.28	1.81	1.90	0.48	0.21	1.42	0.96
$\langle 41, 65, 95, 115|$	1.36	1.29	1.21	1.26	1.21	0.83	0.40	0.74
$\langle 46, 73, 107, 129|$	1.70	1.50	1.46	1.45	0.02	0.14	-0.44	-0.68
$\langle 53, 84, 123, 149|$	1.16	1.06	0.93	0.96	-0.81	-0.62	-0.43	-0.22

$$G(F_9 \cap P_5) = \begin{pmatrix} 1.000 & -0.276 & -0.121 \\ -0.276 & 0.466 & -0.069 \\ -0.121 & -0.069 & 0.121 \end{pmatrix}$$

$$G(F_{10} \cap P_5) = \begin{pmatrix} 1.000 & -0.311 & -0.066 \\ -0.311 & 0.525 & -0.115 \\ -0.066 & -0.115 & 0.164 \end{pmatrix}$$

$$G(F_{12} \cap P_5) = \begin{pmatrix} 1.000 & -0.217 & -0.087 \\ -0.217 & 0.493 & -0.116 \\ -0.087 & -0.116 & 0.159 \end{pmatrix}$$

$$G(F_{16} \cap P_5) = \begin{pmatrix} 1.000 & -0.227 & -0.123 \\ -0.227 & 0.270 & -0.018 \\ -0.123 & -0.018 & 0.104 \end{pmatrix}$$

Really, those off-diagonals aren't going away.

I've already talked about Table 9 in the Pythagorean section. The same comments apply to the 5-limit: the normalized errors are similar but far from identical for different limits with the same mapping. The optimal stretch also differs according to the limit but not as much as for the Pythagorean case. That's probably because there are more intervals being averaged over so the result isn't so sensitive to exactly which ones get included.

Table 10 shows the scalar complexity, badness, and optimal error in the same format as the Tenney limit tables. The different limits still give similar results but, as

with the equal temperaments, the exact figures depend on which limit you look at.

The Tenney-weighted prime-limit measures are significantly smaller than the different Farey limit measures. I use the best normalized formulas I have: Equation 33 for the badness and Equation 32 for the complexity. All that theory does make a difference but it isn't perfect.

4.3 7-Prime Limits

The Farey 7-limit metric is

$$G(F_7) = \begin{pmatrix} 23 & 0 & -4 & -4 \\ 0 & 8 & -2 & -2 \\ -4 & -2 & 6 & -1 \\ -4 & -2 & -1 & 6 \end{pmatrix}$$

Hey, we get a non-zero element! But it won't last. Here's the 8-limit:

$$G(F_8) = \begin{pmatrix} 59 & -3 & -7 & -7 \\ -3 & 9 & -2 & -2 \\ -7 & -2 & 7 & -1 \\ -7 & -2 & -1 & 7 \end{pmatrix}$$

These metrics between them cover the 7-odd limit. If you want to give the factors of 8 less weight you can average

Table 12: Figures for some 7-prime limit rank 2 temperament classes (and the prime limit with Tenney weighting)

Farey limit		7	8	9	10	12	14	15	16	Prime
Name	ETs				Scalar complexity					
Meantone	19&31	1.81	1.88	1.65	1.58	1.62	1.63	1.56	1.53	1.35
Magic	19&41	2.18	2.26	2.09	2.22	2.15	2.07	1.98	2.00	1.80
Orwell	22&31	2.47	2.30	2.59	2.89	2.76	2.65	2.46	2.36	2.26
Miracle	31&41	2.75	2.39	2.90	3.10	3.04	3.00	2.74	2.50	2.44
Garibaldi	12&41	3.05	3.01	2.88	2.78	2.86	2.90	2.73	2.58	2.34
Sensisept	19&27	2.06	2.56	2.42	2.29	2.25	2.11	2.12	2.44	1.89
Pajara	12&22	1.44	1.31	1.45	1.47	1.48	1.49	1.37	1.25	1.20
Myna	27&31	2.15	2.41	2.99	2.88	2.72	2.74	2.85	3.05	2.28
Augene	12&27	1.78	1.58	1.78	1.73	1.72	1.85	1.83	1.66	1.43
Catakleismic	19&53	3.71	3.94	3.40	3.43	3.41	3.35	3.22	3.27	2.87
Name	ETs				Scalar badness (centified)					
Meantone	19&31	2.43	2.38	2.88	2.91	2.83	2.77	2.47	2.43	1.87
Magic	19&41	2.92	2.91	2.52	2.71	2.70	2.51	2.32	2.28	1.93
Orwell	22&31	2.29	1.86	2.28	2.50	2.36	2.49	2.38	2.07	1.69
Miracle	31&41	1.44	1.47	1.81	1.79	1.73	1.66	1.69	1.72	1.26
Garibaldi	12&41	2.84	2.48	2.04	2.19	2.21	2.27	2.27	1.96	1.70
Sensisept	19&27	3.50	3.85	3.45	3.31	3.17	3.26	3.55	3.71	2.50
Pajara	12&22	4.68	4.12	4.14	4.51	4.45	4.40	3.95	3.48	3.08
Myna	27&31	2.49	2.50	3.08	3.08	2.96	2.87	2.94	2.95	2.17
Augene	12&27	4.35	4.55	4.89	4.38	4.32	4.42	4.63	4.78	3.19
Catakleismic	19&53	1.95	2.14	1.89	1.81	1.74	1.78	1.95	2.04	1.38
Name	ETs				Optimal error (cent/oct)					
Meantone	19&31	1.34	1.27	1.74	1.84	1.74	1.69	1.58	1.58	1.38
Magic	19&41	1.34	1.29	1.21	1.22	1.26	1.21	1.17	1.14	1.07
Orwell	22&31	0.93	0.81	0.88	0.86	0.86	0.94	0.97	0.88	0.75
Miracle	31&41	0.52	0.61	0.63	0.58	0.57	0.55	0.62	0.68	0.51
Garibaldi	12&41	0.93	0.82	0.71	0.79	0.77	0.78	0.83	0.76	0.73
Sensisept	19&27	1.70	1.50	1.43	1.44	1.41	1.55	1.67	1.52	1.32
Pajara	12&22	3.25	3.13	2.85	3.06	3.00	2.95	2.88	2.77	2.57
Myna	27&31	1.16	1.04	1.03	1.07	1.09	1.05	1.03	0.97	0.95
Augene	12&27	2.44	2.89	2.74	2.53	2.52	2.39	2.53	2.88	2.23
Catakleismic	19&53	0.52	0.54	0.55	0.53	0.51	0.53	0.60	0.62	0.48

the two metrics out somehow. If you want to allow for more even numbers, maybe you can exaggerate the differences between them.

There are two more Farey limits that naturally fall inside the 7-prime limit.

$$G(F_9) = \begin{pmatrix} 73 & -15 & -7 & -7 \\ -15 & 33 & -4 & -4 \\ -7 & -4 & 8 & -1 \\ -7 & -4 & -1 & 8 \end{pmatrix}$$

$$G(F_{10}) = \begin{pmatrix} 77 & -18 & -3 & -8 \\ -18 & 38 & -7 & -4 \\ -3 & -7 & 12 & -2 \\ -8 & -4 & -2 & 9 \end{pmatrix}$$

They're consistent with the 9-odd limit, along with some more 7-prime Farey limits that I won't give you.

Here are some metrics for further extensions of 7-limit harmony.

$$G(F_{16} \cap P_7) = \begin{pmatrix} 204 & -38 & -21 & -9 \\ -38 & 58 & -1 & -10 \\ -21 & -1 & 22 & -5 \\ -9 & -10 & -5 & 17 \end{pmatrix}$$

$$G(F_{21} \cap P_7) = \begin{pmatrix} 258 & -51 & -11 & -25 \\ -51 & 83 & -9 & -4 \\ -11 & -9 & 30 & -9 \\ -25 & -4 & -9 & 27 \end{pmatrix}$$

Here are some normalized metrics.

$$G(F_7) = \begin{pmatrix} 1.000 & 0.000 & -0.174 & -0.174 \\ 0.000 & 0.348 & -0.087 & -0.087 \\ -0.174 & -0.087 & 0.261 & -0.043 \\ -0.174 & -0.087 & -0.043 & 0.261 \end{pmatrix}$$

$$G(F_8) = \begin{pmatrix} 1.000 & -0.051 & -0.119 & -0.119 \\ -0.051 & 0.153 & -0.034 & -0.034 \\ -0.119 & -0.034 & 0.119 & -0.017 \\ -0.119 & -0.034 & -0.017 & 0.119 \end{pmatrix}$$

$$G(F_9) = \begin{pmatrix} 1.000 & -0.205 & -0.096 & -0.096 \\ -0.205 & 0.452 & -0.055 & -0.055 \\ -0.096 & -0.055 & 0.110 & -0.014 \\ -0.096 & -0.055 & -0.014 & 0.110 \end{pmatrix}$$

$$G(F_{10}) = \begin{pmatrix} 1.000 & -0.234 & -0.039 & -0.104 \\ -0.234 & 0.494 & -0.091 & -0.052 \\ -0.039 & -0.091 & 0.156 & -0.026 \\ -0.104 & -0.052 & -0.026 & 0.117 \end{pmatrix}$$

$$G(F_{16} \cap P_7) = \begin{pmatrix} 1.000 & -0.186 & -0.103 & -0.044 \\ -0.186 & 0.284 & -0.005 & -0.049 \\ -0.103 & -0.005 & 0.108 & -0.025 \\ -0.044 & -0.049 & -0.025 & 0.083 \end{pmatrix}$$

$$G(F_{21} \cap P_7) = \begin{pmatrix} 1.000 & -0.198 & -0.043 & -0.097 \\ -0.198 & 0.322 & -0.035 & -0.016 \\ -0.043 & -0.035 & 0.116 & -0.035 \\ -0.097 & -0.016 & -0.035 & 0.105 \end{pmatrix}$$

Fortunately, the off-diagonals are looking generally smaller than the diagonals now.

Tables 11 and 12 show all the usual numbers for different Farey limits. One thing to note is that the introduction of factors of 9 does make a significant difference to the results, especially for the equal temperaments. It still matters for the rank 2 temperaments—compare the badnesses of meantone, magic and orwell in the 8- and 9-limits. In odd-limit terms this corresponds to the distinction between the 7-limit and the 9-limit. If you want to get octave-equivalent results from an octave-specific formula you could try averaging different metrics to get your one true 7- or 9-limit metric.

4.4 11-Prime Limits

The Farey 11-limit gives the metric

$$G(F_{11}) = \begin{pmatrix} 93 & -17 & -2 & -8 & -8 \\ -17 & 44 & -7 & -4 & -4 \\ -2 & -7 & 14 & -2 & -2 \\ -8 & -4 & -2 & 10 & -1 \\ -8 & -4 & -2 & -1 & 10 \end{pmatrix}$$

The 12-limit gives you a bit more harmonic freedom.

$$G(F_{12}) = \begin{pmatrix} 109 & -9 & -4 & -10 & -10 \\ -9 & 48 & -8 & -5 & -5 \\ -4 & -8 & 15 & -2 & -2 \\ -10 & -5 & -2 & 11 & -1 \\ -10 & -5 & -2 & -1 & 11 \end{pmatrix}$$

After that, of course, factors of 13 come into the picture. So you can think of these two metrics as defining the 11-odd limit, give or take a few generous inversions. However, the 11-prime subset of the 14-limit is still within the 11-odd limit.

$$G(F_{14} \cap P_{11}) = \begin{pmatrix} 114 & -12 & -5 & -5 & -11 \\ -12 & 53 & -8 & -8 & -5 \\ -5 & -8 & 16 & -3 & -2 \\ -5 & -8 & -3 & 16 & -2 \\ -11 & -5 & -2 & -2 & 12 \end{pmatrix}$$

Some more metrics for extended 11-limit harmony

$$G(F_{15} \cap P_{11}) = \begin{pmatrix} 129 & -19 & -12 & -4 & -11 \\ -19 & 60 & -1 & -10 & -6 \\ -12 & -1 & 23 & -5 & -3 \\ -4 & -10 & -5 & 18 & -2 \\ -11 & -6 & -3 & -2 & 13 \end{pmatrix}$$

$$G(F_{16} \cap P_{11}) = \begin{pmatrix} 241 & -35 & -20 & -8 & -15 \\ -35 & 66 & 0 & -10 & -6 \\ -20 & 0 & 25 & -5 & -3 \\ -8 & -10 & -5 & 19 & -2 \\ -15 & -6 & -3 & -2 & 14 \end{pmatrix}$$

$$G(F_{21} \cap P_{11}) = \begin{pmatrix} 300 & -46 & -8 & -24 & -18 \\ -46 & 96 & -8 & -3 & -9 \\ -8 & -8 & 34 & -9 & -4 \\ -24 & -3 & -9 & 30 & -3 \\ -18 & -9 & -4 & -3 & 17 \end{pmatrix}$$

$$G(F_{32} \cap P_{11}) = \begin{pmatrix} 712 & -127 & -73 & -22 & -20 \\ -127 & 271 & -36 & -15 & -22 \\ -73 & -36 & 123 & -20 & -11 \\ -22 & -15 & -20 & 49 & -6 \\ -20 & -22 & -11 & -6 & 32 \end{pmatrix}$$

Because the matrices are getting very wide, I'll use a different format for the normalized metrics.

$$F_{11} = \begin{pmatrix} 1.000 & -0.183 & -0.022 & -0.086 & -0.086 \\ -0.183 & 0.473 & -0.075 & -0.043 & -0.043 \\ -0.022 & -0.075 & 0.151 & -0.022 & -0.022 \\ -0.086 & -0.043 & -0.022 & 0.108 & -0.011 \\ -0.086 & -0.043 & -0.022 & -0.011 & 0.108 \end{pmatrix}$$

Table 13: Errors and the like for some 11-prime limit equal temperaments

Farey limit	11	12	14	15	16	11	12	14	15	16
Mapping	Pure octave RMS (cent)					Optimal RMS (cent)				
$\langle 12, 19, 28, 34, 42|$	27.24	27.62	27.66	26.79	26.40	23.64	25.18	24.58	24.16	24.63
$\langle 14, 22, 32, 39, 48|$	25.93	25.43	24.63	28.83	29.63	19.92	20.55	20.22	22.33	26.03
$\langle 15, 24, 35, 42, 52|$	19.26	19.07	20.00	22.10	21.94	18.40	17.99	19.66	21.02	21.32
$\langle 19, 30, 44, 53, 65|$	22.23	22.12	21.70	21.14	21.13	18.27	18.96	18.41	18.08	19.24
$\langle 19, 30, 44, 53, 66|$	15.77	15.66	16.39	16.43	16.14	15.76	15.65	16.23	16.07	15.95
$\langle 22, 35, 51, 62, 76|$	10.91	10.82	11.08	10.64	10.41	10.88	10.75	10.86	10.44	10.30
$\langle 27, 43, 63, 76, 94|$	13.39	13.13	12.85	13.73	14.06	9.92	10.11	10.44	10.57	12.23
$\langle 31, 49, 72, 87, 107|$	6.47	6.35	6.31	6.09	6.07	5.87	5.69	5.87	5.59	5.77
$\langle 41, 65, 95, 115, 142|$	4.70	4.66	4.65	4.79	4.68	4.70	4.65	4.65	4.76	4.66
$\langle 46, 73, 107, 129, 159|$	4.78	4.76	4.93	5.61	5.49	4.78	4.76	4.91	5.58	5.47
Mapping	Normalized optimum (cent/oct)					Octave stretch (cent)				
$\langle 12, 19, 28, 34, 42|$	7.43	7.84	7.55	7.31	7.10	-8.38	-6.89	-7.46	-6.62	-5.36
$\langle 14, 22, 32, 39, 48|$	6.26	6.40	6.21	6.76	7.50	10.44	9.22	8.38	10.58	8.07
$\langle 15, 24, 35, 42, 52|$	5.79	5.60	6.04	6.36	6.14	-3.53	-3.86	-2.16	-3.92	-2.94
$\langle 19, 30, 44, 53, 65|$	5.75	5.90	5.65	5.47	5.54	7.94	7.01	6.83	6.34	4.98
$\langle 19, 30, 44, 53, 66|$	4.96	4.87	4.98	4.86	4.60	0.25	0.36	1.36	1.97	1.40
$\langle 22, 35, 51, 62, 76|$	3.42	3.35	3.33	3.16	2.97	-0.50	-0.75	-1.30	-1.17	-0.84
$\langle 27, 43, 63, 76, 94|$	3.12	3.15	3.21	3.20	3.52	-5.58	-5.10	-4.41	-5.02	-3.92
$\langle 31, 49, 72, 87, 107|$	1.85	1.77	1.80	1.69	1.66	1.70	1.72	1.38	1.40	1.07
$\langle 41, 65, 95, 115, 142|$	1.48	1.45	1.43	1.44	1.34	-0.10	-0.19	-0.02	0.32	0.23
$\langle 46, 73, 107, 129, 159|$	1.50	1.48	1.51	1.69	1.58	0.02	-0.07	0.26	-0.31	-0.20

$$F_{12} = \begin{pmatrix} 1.000 & -0.083 & -0.037 & -0.092 & -0.092 \\ -0.083 & 0.440 & -0.073 & -0.046 & -0.046 \\ -0.037 & -0.073 & 0.138 & -0.018 & -0.018 \\ -0.092 & -0.046 & -0.018 & 0.101 & -0.009 \\ -0.092 & -0.046 & -0.018 & -0.009 & 0.101 \end{pmatrix}$$

$$F_{21} : \begin{pmatrix} 1.000 & -0.153 & -0.027 & -0.080 & -0.060 \\ -0.153 & 0.320 & -0.027 & -0.010 & -0.030 \\ -0.027 & -0.027 & 0.113 & -0.030 & -0.013 \\ -0.080 & -0.010 & -0.030 & 0.100 & -0.010 \\ -0.060 & -0.030 & -0.013 & -0.010 & 0.057 \end{pmatrix}$$

$$F_{14} : \begin{pmatrix} 1.000 & -0.105 & -0.044 & -0.044 & -0.096 \\ -0.105 & 0.465 & -0.070 & -0.070 & -0.044 \\ -0.044 & -0.070 & 0.140 & -0.026 & -0.018 \\ -0.044 & -0.070 & -0.026 & 0.140 & -0.018 \\ -0.096 & -0.044 & -0.018 & -0.018 & 0.105 \end{pmatrix}$$

$$F_{32} : \begin{pmatrix} 1.000 & -0.178 & -0.103 & -0.031 & -0.028 \\ -0.178 & 0.381 & -0.051 & -0.021 & -0.031 \\ -0.103 & -0.051 & 0.173 & -0.028 & -0.015 \\ -0.031 & -0.021 & -0.028 & 0.069 & -0.008 \\ -0.028 & -0.031 & -0.015 & -0.008 & 0.045 \end{pmatrix}$$

$$F_{15} : \begin{pmatrix} 1.000 & -0.147 & -0.093 & -0.031 & -0.085 \\ -0.147 & 0.465 & -0.008 & -0.078 & -0.047 \\ -0.093 & -0.008 & 0.178 & -0.039 & -0.023 \\ -0.031 & -0.078 & -0.039 & 0.140 & -0.016 \\ -0.085 & -0.047 & -0.023 & -0.016 & 0.101 \end{pmatrix}$$

As with all the other limits, there are tables for the 11-prime Farey limits. This time they're numbered 13 and 14. They show the trends observed in previous limits. The different Farey limits are getting more consistent (probably because there are always plenty of numbers to average over) but the normalized Farey-limit errors are significantly larger than the Tenney-weighted prime-limit errors. That means the normalization isn't quite getting the balance right but it still does a much better job than anything else I can think up.

$$F_{16} : \begin{pmatrix} 1.000 & -0.145 & -0.083 & -0.033 & -0.062 \\ -0.145 & 0.274 & 0.000 & -0.041 & -0.025 \\ -0.083 & 0.000 & 0.104 & -0.021 & -0.012 \\ -0.033 & -0.041 & -0.021 & 0.079 & -0.008 \\ -0.062 & -0.025 & -0.012 & -0.008 & 0.058 \end{pmatrix}$$

Table 14: Figures for some 11-prime limit rank 2 temperament classes (and the prime limit with Tenney weighting)

	Farey limit	11	12	14	15	16	18	Prime
Name	ETs			Scalar complexity				
Orwell	22&31	2.51	2.42	2.33	2.18	2.11	2.14	2.05
Meantone	12&31	2.41	2.53	2.42	2.35	2.33	2.35	1.92
Miracle	31&41	3.32	3.21	3.24	3.08	2.89	2.88	2.79
Valentine	15&31	2.74	2.64	2.84	2.90	2.83	2.85	2.31
Myna	27&31	3.20	3.14	3.23	3.15	3.39	3.28	2.61
Magic	22&41	2.75	2.70	2.74	2.63	2.49	2.49	2.35
Squares	14&31	2.27	2.33	2.23	2.40	2.57	2.51	2.20
Meanpop	19a&31	2.77	2.73	2.85	2.77	2.57	2.54	2.42
Pajara	12&22	2.03	2.12	2.05	1.95	1.84	1.93	1.61
Augene	12&15	1.76	1.80	1.81	1.83	1.69	1.76	1.45
Name	ETs			Scalar badness (centified)				
Orwell	22&31	3.35	3.18	3.34	2.99	2.73	2.73	2.36
Meantone	12&31	4.28	4.32	4.23	3.85	3.65	3.77	2.76
Miracle	31&41	1.82	1.69	1.65	1.76	1.80	1.75	1.35
Valentine	15&31	3.29	3.11	3.22	3.39	3.10	3.06	2.39
Myna	27&31	2.99	3.02	2.99	2.83	2.90	2.89	2.22
Magic	22&41	3.79	3.64	3.58	3.42	3.00	3.00	2.88
Squares	14&31	4.12	4.04	3.97	4.04	4.26	4.16	3.19
Meanpop	19a&31	4.38	4.10	4.15	3.79	3.63	3.60	2.99
Pajara	12&22	5.31	5.42	5.23	4.87	4.52	4.62	3.71
Augene	12&15	5.48	5.55	5.78	5.84	5.96	5.99	3.84
Name	ETs			Optimal error (cent/oct)				
Orwell	22&31	1.33	1.31	1.43	1.37	1.30	1.28	1.15
Meantone	12&31	1.77	1.71	1.75	1.64	1.57	1.60	1.44
Miracle	31&41	0.55	0.53	0.51	0.57	0.62	0.61	0.48
Valentine	15&31	1.20	1.18	1.13	1.17	1.10	1.07	1.03
Myna	27&31	0.94	0.96	0.92	0.90	0.86	0.88	0.85
Magic	22&41	1.38	1.35	1.30	1.30	1.20	1.21	1.23
Squares	14&31	1.82	1.74	1.78	1.68	1.66	1.65	1.45
Meanpop	19a&31	1.58	1.50	1.45	1.37	1.41	1.42	1.24
Pajara	12&22	2.61	2.56	2.56	2.50	2.46	2.40	2.30
Augene	12&15	3.11	3.09	3.20	3.20	3.52	3.41	2.65

5 Conclusion

The Tenney-limit results are so close to the Tenney-weighted prime-limit results that you may as well go with the latter. Then again, as a Tenney-limit is no harder to implement than an arbitrary prime weighting you may as well go with what you believe in. The Tenney limits do tend to include some uselessly large intervals which is why they manage to give such a simple metric.

Farey limits are probably a more realistic approximation to the intervals you might want to use. It does make a difference which limit you choose here so if you know what limit you want it's best to specify it. It isn't much harder to do the calculation for an arbitrary metric compared to prime weights.

If you don't know exactly what limit you want you can always average out the metrics from different limits. If you want some larger intervals than the Farey limits give you can add some Tenney metrics to the mix. The cross-weighting term I added to the prime weighted metrics will control how much you concentrate on small intervals relative to a Tenney limit. So you could try playing with that if you'd prefer to start with a Tenney limit.

Ultimately you can get whatever you want by giving exactly the weights you want to exactly the intervals you're interested in. Once you set up the problem it's easy to solve for any metric. But Tenney weighting of prime limits is still easier to set up than the general cases so if you don't

know what you want you may as well use that (and remember to take account of the optimal scale stretch when you calculate the errors).

6 References

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7 Glossary

Error The deviation of an interval from just intonation. Sometimes the absolute value, sometimes weighted.

Farey limit The Farey f-limit F_f contains all ratios $n{:}d$ in their lowest terms where $1 <= d <= n <= f$.

Mapping "The mapping" of an equal temperament is a column vector containing numbers telling you how many scale steps there are to each prime interval. The mapping of a rank 2 temperament is a matrix where each column is the mapping of an equal temperament.

Metric A matrix that tell you how complexity and error should be measured.

Normalization Multiplying an average by a function of the things you're averaging over, but not the thing you're averaging, so that averages over different things give similar results. Or multiplying by some other constant that doesn't affect the result.

Optimal error The normalized, optimal, RMS error over a given set of intervals.

Prime interval Intervals of minimal complexity that all the intervals you're interested in are built from. For all the examples here that means ratios of the form $p{:}1$ where p is a prime number.

Prime limit The p-prime limit P_p contains all ratios $n{:}d$ in their lowest terms where $1 <= d <= n$ and n and d contain no prime factors larger than p.

Product limit The same as a Tenney limit the way I specify it today.

Regular temperament A way of simplifying just intonation so that it requires fewer distinct intervals but loses accuracy. Each interval in just intonation corresponds to a single tempered interval, but a tempered interval will correspond to more than one just interval.

Renormalization Nothing to be frightened of! All I mean is a second stage of normalization. Nothing to do with the same term used in quantum field theory.

RMS Root mean squared.

Scalar complexity A way of measuring the complexity of a temperament class. See *Primerr* for details. Related to the RMS of the numbers of notes you need for the intervals under consideration.

Scalar badness Scalar complexity times optimal error.

Temperament class Different regular temperaments with the same mapping from just to tempered intervals. Also a hazy generalization that includes irregular temperaments.

Tenney limit The Tenney t-limit T_t contains all ratios $n{:}d$ in their lowest terms where $1 <= d <= n$ and $nd <= t$. It can be specified different ways.

Tenney weighting Giving each prime interval a weight inversely proportional to its size in octaves or cents.

TOP-RMS Tenney-weighted Optimal Prime RMS.